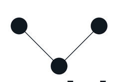
Periodicity and Homomesy for the $\mathsf{V} \times [n]$ poset

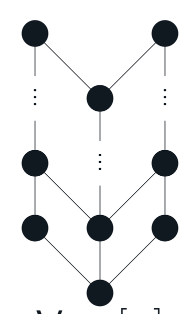
Matthew Plante
Department of Mathematics, University of Connecticut

Abstract

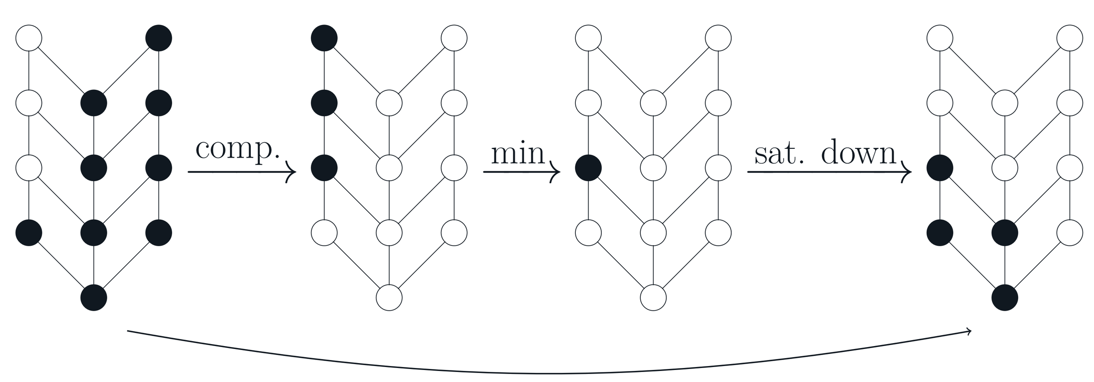
The poset $\mathsf{V} \times [n]$, the Cartesian product of a three-element $\vee$-shaped poset with a chain of length [n] has recently emerged as an example of interest in Dynamical Algebraic Combinatorics. Most posets that are known to have "nice" small-order periodicity for rowmotion arise either from representation theory as root or minuscule posets, or are built up inductively in a simple way ("skeletal posets" as defined by Grinberg & Roby). Here we show that the order of rowmotion of $\mathsf{V} \times [n]$ is 2(n+2), and prove several general homomesies for it.

Rowmotion on $\mathsf{V} \times [n]$

- Let V be the three-element poset 
- The poset of interest is $V(n) = \mathsf{V} \times [n]$
- Let $\mathcal{J}_n$ denote the set of order ideals of $V(n)$. That is, for $I \subseteq V(n)$, $I \in \mathcal{J}_n \iff x \in I \implies y \in I$ for all $y < x$.
- Denote *rowmotion* on order ideals by Row. We compose Row on order ideals by taking the minimal elements of the complement and saturating down.

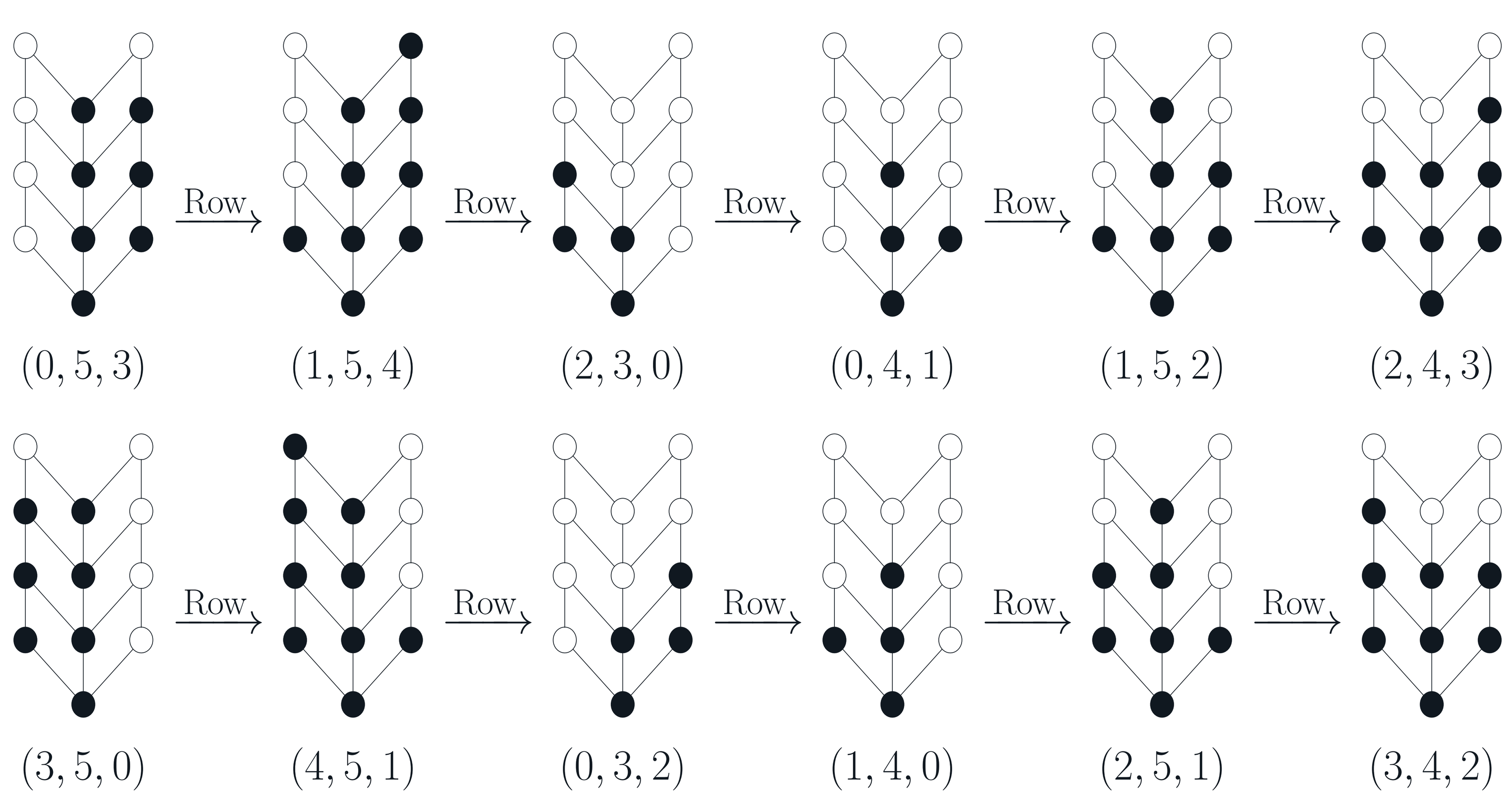


$\mathsf{V} \times [n]$



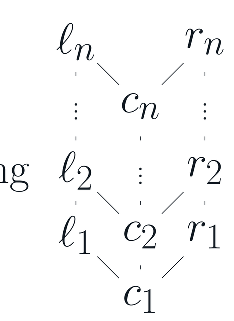
Example Orbit of Length $2(n+2)$

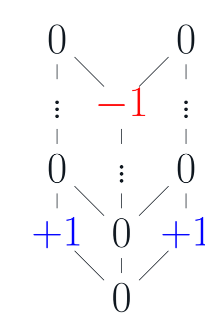
Theorem 1. *Order ideals of $V(n)$ are reflected about the center chain after $n+2$ iterations of* Row*, and furthermore, the order of* Row *on order ideals of $V(n)$ is* $2(n{+}2)$.

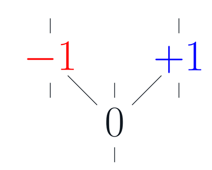


Homomesy for $\mathsf{V} \times [n]$

Definition 1 (PR15, Def. 1.1)**.** *If S is a set and τ an invertible action on S then we say a statistic $f : S \to K$ is* homomesic *if there exists $c \in K$ such that $\frac{\Sigma_{s \in O} f(s)}{\# O} = c$ for all orbits O of τ. When this holds, we also say f is* c-mesic.

Using this labeling  we get the following theorem.



Theorem 2. *For $\mathcal{J}_n$ with* Row, *$\chi_{\ell_1} + \chi_{r_1} - \chi_{c_n}$ is $\frac{2(n-1)}{n+2}$-mesic and $\chi_{r_i} - \chi_{\ell_i}$ is 0-mesic*  *, for each $i \in [n]$, where χ_x is the indicator function.*

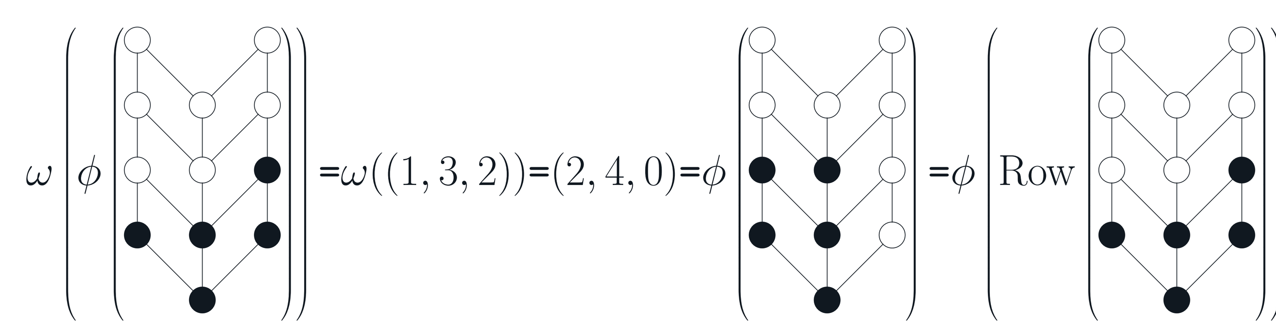
Bijecting to Triples

Definition 2. *1. Denote $T_n = \{(a,b,c) \in \{0,\ldots,n+1\}^3 : a, c < b\}$.*

2. Define $\phi : \mathcal{J}_n \to T_n$ by $\phi(I) = (\Sigma\chi_{\ell_i}, 1 + \Sigma\chi_{c_i}, \Sigma\chi_{r_i})$.

Definition 3. *Define the action ω on $(a,b,c) \in T_n$ as the process:*
1. $a \to a+1$ unless $b = a+1$, then $a \to 0$.
2. Repeat step 1 with c instead of a.
3. $b \to b+1$ unless $b = n+1$, then $b \to \max(a,c)+1$.

Proposition 1. *The map ϕ is an equivariant bijection that sends* Row *to ω.*



References

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[PR15] James Propp and Tom Roby, *Homomesy in products of two chains*, Electronic J. Combin. **22(3)** (2015), #P3.4, `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p4`.

[R16] Tom Roby, *Dynamical algebraic combinatorics and the homomesy phenomenon* in A. Beveridge, et. al., Recent Trends in Combinatorics, IMA Volumes in Math. and its Appl., **159** (2016), 619–652.

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Sketch of Proof of Periodicity

One Case: Iterate ω on an element of the form $(0, k, j)$, with $j \geq 1$, and $k - j \geq 3$.

0	k	j	
$\vdots$	$\vdots$	$\vdots$	ω^{n-k+1}
$n-k+1$	$n+1$	$n-k+j+1$	
$n-k+2$	$n-k+j+3$	$n-k+j+2$	
$n-k+3$	$n-k+j+4$	0	
$\vdots$	$\vdots$	$\vdots$	ω^{k-j-3}
$n-j$	$n+1$	$k-j-3$	
$n-j+1$	$n-j+2$	$k-j-2$	
0	$n-j+3$	$k-j-1$	
$\vdots$	$\vdots$	$\vdots$	ω^{j-2}
$j-2$	$n+1$	$k-3$	
$j-1$	$k-1$	$k-2$	
j	k	0	

Counting the number of times we iterated ω we get

$$(n-k+1) + (2) + (k-j-3) + (2) + (j-2) + (2) = n+2.$$

Other cases are similar. After iterating ω on all elements of the form $(0,k,j)$ for any j and k we verify $\omega^{n+2}(0,k,j) = (j,k,0)$. Since $\omega^{-a}(a,b,c) = (0,k,j)$, we can compose $\omega^a \omega^{n+2} \omega^{-a}(a,b,c) = (c,b,a)$.

Center-Seeking Snakes

Here we see a decomposition of the orbit board of $(1,3,2) \in T_4$ into 6 snakes $0, 1, \ldots, n+1$. These snakes start on the left and/or right lanes and "move" into the center lane. We call these snakes, *center-seeking snakes*.

1 3 2
2 4 0
3 5 1
4 5 2
0 4 3
1 5 0
2 3 1
0 4 2
1 5 3
2 5 4
3 4 0
0 5 1

0 3 0
1 4 1
2 5 2
3 4 3
0 5 0
1 2 1

Similarly we get 2 two-tailed center-seeking snakes in the orbit board of $(0,3,0) \in T_4$. This phenomenon is typical of symmetric triples.

For (a,b,c) if $a > 0$ then $\chi_{\ell_1} = 1$ and if $c > 0$ then $\chi_{r_1} = 1$. Also the only time $\chi_{c_n} = 1$ is when $b = n+1$. Since there are 6 snakes that start with 0 and end with $n+1$ we get $\chi_{\ell_1} + \chi_{r_1} - \chi_{c_n} = 4(n+2) - 6 - 6$; or there are 2 snakes with 4 tails so $\chi_{\ell_1} + \chi_{r_1} - \chi_{c_n} = 2(n+2) - 4 - 2$ and thus

$$\frac{4(n+2)-12}{2(n+2)} = \frac{2(n+2)-6}{n+2} = \frac{2n-2}{n+2}.$$

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