Periodicity and Homomesy for Whirling Proper 3-Colorings of a Graph

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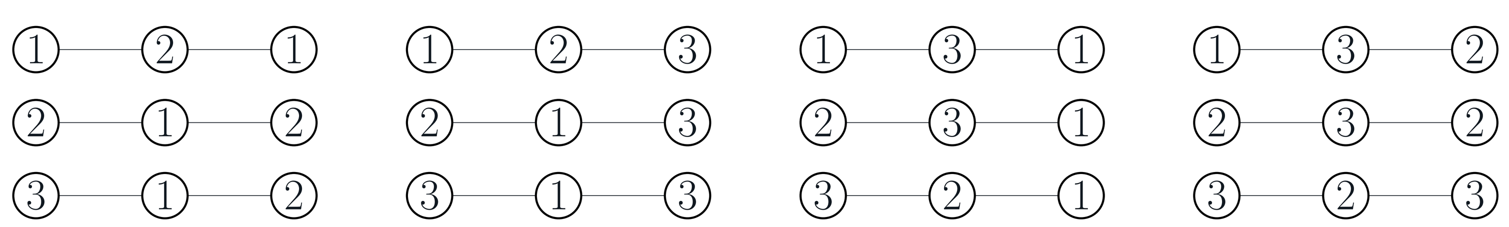
Abstract

A proper k-coloring of a graph is a labeling of the vertices with $1, \dots, k$ where no two adjacent vertices have the same label. We define a periodic action on the set of all proper k-colorings of a graph. The action is a product of whirls at each vertex, (which can also be thought as a generalization of the action of toggling independent sets,) defined by cyclically incrementing a vertex label until the result is again a proper k-coloring. Here we show results on the periodicity and general homomesies of the action on proper 3-colorings of both path graphs and cycle graphs.

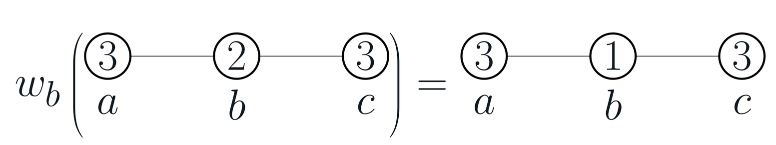
The action ω on $K_k(G)$

- Let $G = (V, E)$ be a graph with $K_k(G)$ being the set of proper k-colorings $\kappa : V \to [k]$.

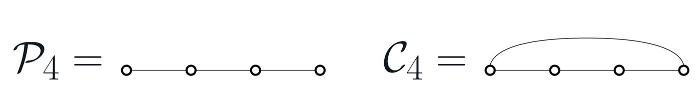
All Proper 3-Colorings of



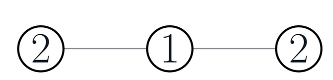
Definition 1 (JPR18, Def 2.1)**.** *Define $w_v : K(G) \to K(G)$ (whirl at v) by incrementing the color of vertex v by 1 modulo k repeatedly until arriving at a proper k-coloring.*



- Let $\mathcal{P}_n$ be the path graph with n vertices, and let $\mathcal{C}_n$ be the cycle graph with n vertices.



- We set $V = [n]$ labeled from left to right and consider the action $\omega = w_n \dots w_1$. Thus the proper k-colorings of $\mathcal{P}_n$ are maps $\kappa : [n] \to [k]$ such that $\kappa(i-1) \neq \kappa(i) \neq \kappa(i+1)$ (modulo n if we are on a cyclic graph.) We also represent colorings with $[k]$-words of length n.

 $\to 212$

$$\omega(212) = w_3 w_2 w_1(212) = w_3 w_2(312) = w_3(312) = 313$$

Homomesy for ω acting on $K_3(\mathcal{P}_n)$ and $K_3(\mathcal{C}_n)$

Definition 2 (PR15, Def. 1.1)**.** *Let S be a set and τ an invertible action on S. We say a statistic $f : S \to K$ is* homomesic *if there exists $c \in K$ such that $\frac{\Sigma_{s \in O} f(s)}{\# O} = c$ for all orbits O of τ. When this holds, we also say f is c-mesic.*

Theorem 1. *Fix any color $j \in [3]$. Set $\chi_i := \chi_{i,j}$ ($\chi_i(\kappa)$ is 1 if $\kappa(i) = j$ and 0 otherwise). Under the action of ω on $K_3(\mathcal{P}_n)$,*

1. *$\chi_i - \chi_{n+1-i}$ is 0-mesic, and*
2. *$2\chi_1 + \chi_2$ is 1-mesic and $\chi_{n-1} + 2\chi_n$ is 1-mesic.*

Theorem 2. *Fix any color $j \in [3]$. Set $\chi_i := \chi_{i,j}$. Under the action of ω on $K_3(\mathcal{C}_n)$,*

1. *If $3 \nmid n$, then χ_i is 1/3-mesic, and*
2. *If $3|n$, then $\chi_{3a+i} - \chi_{3b+i}$ is 0-mesic for $i \in [3]$ and $0 \leq a, b \leq \frac{n}{3} - 1$.*

Difference Vector

Definition 3. *The* difference vector, *d of a proper 3-coloring of $\mathcal{P}_n$ is the string of $n-1$ $+$'s and $-$'s depending on whether the coloring increases by 1 or decreases by 1 respectively from left to right.*

1 2 3 2	+ + −
3 1 3 1	+ − +
2 1 2 3	− + +
3 1 2 1	+ + −
2 3 2 3	+ − +
1 3 1 2	− + +
2 3 1 3	+ + −
1 2 1 2	+ − +
3 2 3 1	− + +

Similarly the difference vector, d, of a proper 3-coloring of $\mathcal{C}_n$ is the same as the difference vector of $\mathcal{P}_n$ but with an extra $+$ or $-$ for the difference between the last color and the first color

3 2 1 2 1   − − + − −

We set $s_i(d)$ to be $\#+ - \#-$ modulo k up to the ith entry in d, and call $s_n(d)$ the 'sum of d'. In the last example, the sum of $- - + - -$ is 0.

The affect of ω on difference vectors

Lemma 1. *If $\kappa \in K_3(\mathcal{P}_n)$ or $\kappa \in K_3(\mathcal{C}_n)$ with difference vector d,*

1. *If i is an interior vertex (degree two), then the difference vector of $w_i(\kappa)$ is d but with d_{i-1} and d_i swapped.*
2. *If i is an exterior vertex (degree one) and $i = 1$ (resp. $i = n$), then the difference vector of $w_i(\kappa)$ is d but with d_1 (resp. d_{n-1}) changed from $+$ to $-$ or vise versa.*

Here is an example where ω acts on κ one whirl at a time with the difference vector updated at each step.

	1 2 1 2 3 1	+ − + + +
w_1	3 2 1 2 3 1	− − + + +
w_2	3 2 1 2 3 1	− − + + +
w_3	3 2 3 2 3 1	− + − + +
w_4	3 2 3 1 3 1	− + + − +
w_5	3 2 3 1 2 1	− + + + −
w_6	3 2 3 1 2 3	− + + + +

References

[PR15] James Propp and Tom Roby, *Homomesy in products of two chains*, Electronic J. Combin. **22(3)** (2015), #P3.4, `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p4`.

[JPR18] Michael Joseph and James Propp and Tom Roby, *Whirling injections, surjections, and other functions between finite sets*, 2018. `https://arxiv.org/abs/1711.02411`

Periodicity of ω on proper colorings

Proposition 1. *If $\kappa \in K_3(\mathcal{P}_n)$, d is the difference vector of κ, and τ is leftward cyclic-shift on strings of $+$'s and $-$'s, then the difference vector $\omega(\kappa)$ is $\tau(d)$.*

$$1\,2\,1\,3\,1\,2 \xrightarrow{\omega} 3\,2\,1\,2\,3\,1$$
$$+ - - + + \xrightarrow{\tau} - - + + +$$

Proposition 2. *If $\kappa \in K_3(\mathcal{C}_n)$, d is the difference vector of κ, and τ' is leftward cyclic-shift on the first $n-1$ elements of strings of $+$'s and $-$'s, then the difference vector $\omega(\kappa)$ is $\tau'(d)$.*

$$1\,2\,1\,2 \xrightarrow{\omega} 3\,2\,3\,1$$
$$+ - + - \xrightarrow{\tau'} - + + -$$

Theorem 3. *Let $\kappa \in K_3(\mathcal{P}_n)$ have difference vector d. Let ℓ be the smallest natural number such that $\omega^\ell(\kappa) = \kappa$, t be the smallest natural number such that $\tau^t(d) = d$.*

1. *If the sum of d is 0, then $\ell = t$.*
2. *Otherwise $\ell = 3t$.*

Sketch of Proof of Homomesy

Let ℓ be the smallest natural number such that $\omega^\ell(\kappa) = \kappa$, and let t be the smallest natural number such that $\tau^t(d) = d$.

If $\ell = 3t$, then the orbit contains the other two proper 3-colorings with difference vector d, therefore every color appears in the each spot exactly 1/3 of the time. So we will focus on the case where $\ell = t$. To be precise, we will show

$$\kappa(i) = \omega^{i-1}\kappa(n+1-i)$$

completing the proof.

1	3	2	3	1	2	1
2	1	2	3	1	3	2
3	1	2	3	2	1	3
2	3	1	3	2	1	2
1	2	1	3	2	3	1
3	2	1	3	1	2	3

If the first entry in the difference vector is $+$, then we subtract one from the first color in κ. Recall $s_i(d)$ is the partial sum of the first i entries in d, that is, $\#+ - \#-$ modulo 3 up to the ith entry in d. Since the sum of d is 0, we know $s_n(d) \equiv 0$ modulo 3. It follows that $\kappa(i) = \kappa(1) + s_{i-1}(d)$ and $\omega^{i-1}\kappa(1) = \kappa(1) - s_i(d)$. Therefore

$$\omega^{i-1}\kappa(n-i+1) = \kappa(1) - s_i(d) + s_{n-i}(\tau^{i-1}d)$$

But $s_n(d) \equiv 0 \mod 3$ so

$$s_{n-i}(\tau^{i-1}d) - s_i(d) \equiv s_{n-i}(\tau^{i-1}d) + s_i(d) - s_i(d) - s_i(d) \equiv -2s_i(d) \equiv s_i(d).$$

So $\kappa(i) = \omega^{i-1}\kappa(n+1-i)$. The argument is similar for $\kappa \in K_3(\mathcal{C}_n)$.

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