Whirling P-partitions and rowmotion on chain-factor posets

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ABSTRACT

We investigate the dynamics of certain natural actions on labelings of partially ordered sets (posets) and related objections. Rowmotion (Definition 2.1.5) is an invertible map on order ideals of a poset which has received much attention recently from researchers in dynamical algebraic combinatorics. Of particular interest are the order of the rowmotion map and the homomesy phenomenon (Definition 1.1.2). In this dissertation we look at rowmotion on order ideals of posets of families not yet thoroughly investigated, including *fence posets*, obtained by arbitrarily ordering each edge in a path graph up or down. These posets are important in the theory of cluster algebras and q-analogues. The rowmotion orbits of antichains of fence posets can be succinctly visualized using certain tilings of a cylinder.

Another family is the $\vee \times [k]$ posets, where $\vee$ is the $\vee$-shaped three element poset with two relations and one minimal element, and $[k]$ is a k-element chain. Here we make an equivariant bijection between rowmotion on order ideals and the whirling action, introduced by Joseph, Propp, and Roby, on P-partitions of $\vee$. Finally we investigate whirling proper colorings of path graphs and cycle graphs. This is motivated as a generalization of toggling independent sets of a path graph, which itself has an equivariant bijection to rowmotion on order ideals of a special type of fence poset.

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Contents

Chapter 1

Introduction

In most settings of dynamical algebraic combinatorics a map τ acts on a set of combinatorial objects. When τ is invertible, the set is partitioned into orbits. Questions concerning periodicity and orbit structure arise and are resolved with a combination of algebraic and bijective methods. The *homomesy phenomenon* (Definition 1.1.2) is of particular interest in the field [8, 1, 12] and has become a focus of the author in several projects. this thesis will explore these themes of dynamical algebraic combinatorics for previously unstudied sets of combinatorial objects and invertible maps τ, as well as briefly expositing earlier related work.

After giving a formal definition of homomesy with examples, we will move on to the main combinatorial setting: antichains and order ideals of partially ordered sets. In Section 2 we define the action of *rowmotion*, stating old and new results on periodicity and homomesy on a poset. One useful strategy in proving results in this thesis is to find *equivariant bijections*, which carry an action on one set to a different action on another set. In Section 3 we define *whirling* on P-partitions of a partially ordered set. In this section we will also establish an equivariant bijection between order-reversing labelings of a poset P and rowmotion on $P \times [k]$. In Section 4 we show a new result on periodicity and some homomesies regarding rowmotion on $\mathsf{V} \times [k]$, using the equivariant bijection from Section 3. Finally, in Section 5 we will give a generalization of independent sets on a graph, namely partial proper colorings, and prove a result for graphs of the form $K_n \times [n]$.

1.1 Homomesy

To motivate homomesy we start with an example where the combinatorial setting is binary strings and the action is leftward cyclic shift. We define $[n] = \{1, 2, \ldots, n-1, n\}$ and $[m, n] = \{m, m+1, \ldots, n-1, n\}$. In particular we will use $[0, n] = \{0, 1, 2, \ldots, n\}$.

Example 1.1.1. Let $\mathcal{S} \subset \{0,1\}^4$ be the set of binary strings of length 4 with exactly 2 zeros, and let τ be the action of leftward cyclic shift on binary strings. The group action $\langle \tau \rangle$ partitions $\mathcal{S}$ into one orbit of length 4 and one of length 2.

$$1001 \xrightarrow{\tau} 0011 \xrightarrow{\tau} 0110 \xrightarrow{\tau} 1100 \xrightarrow{\tau}$$

$$0101 \xrightarrow{\tau} 1010 \xrightarrow{\tau}$$

We borrow from the symmetric group the inversions statistic inv which returns the number of times a 0 comes after a 1 in the binary string. This statistic is 2-mesic.

$$\mathrm{inv}(1001) + \mathrm{inv}(0011) + \mathrm{inv}(0110) + \mathrm{inv}(1100) = 2 + 0 + 2 + 4 = 8.$$

$$\mathrm{inv}(0101) + \mathrm{inv}(1010) = 1 + 3 = 4.$$

So in either case we get that the sum of the inversion statistic over an orbit is twice the orbit size. This phenomenon of a statistic having the same average over all orbits is known as *homomesy.*

Definition 1.1.2 ([27])**.** Given a set $\mathcal{S}$, an invertible map $\tau : \mathcal{S} \to \mathcal{S}$ such that each τ-orbit is finite, and a statistic $f : \mathcal{S} \to \mathbb{K}$ (for some field $\mathbb{K}$ of characteristic zero), we say the triple $(\mathcal{S}, \tau, f)$ exhibits *homomesy* if there exists a constant $c \in \mathbb{K}$ such that for every τ-orbit $\mathcal{O} \subset \mathcal{S}$

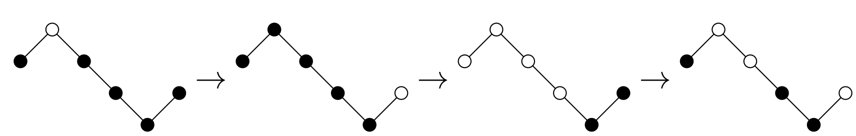
$$\frac{\sum_{x\in\mathcal{O}} f(x)}{\#\mathcal{O}} = c.$$

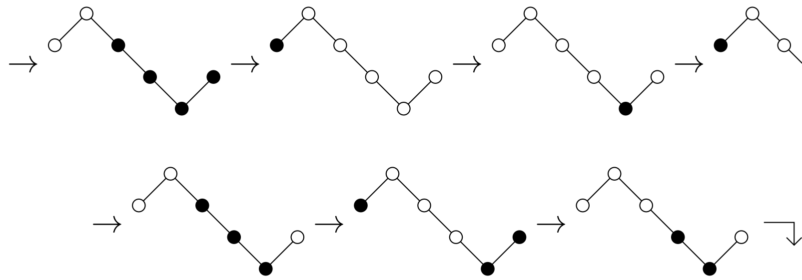
In this situation we say that the function f is *homomesic* under the action of the cyclic group $\langle\tau\rangle$ on $\mathcal{S}$, or more specifically *c-mesic.*

Proposition 1.1.3 ([27])**.** *Let τ be leftward cyclic shift on binary strings of length $a+b$ with exactly a 1's. Then* $\mathrm{inv}(s)$ *is $ab/2$-mesic.*

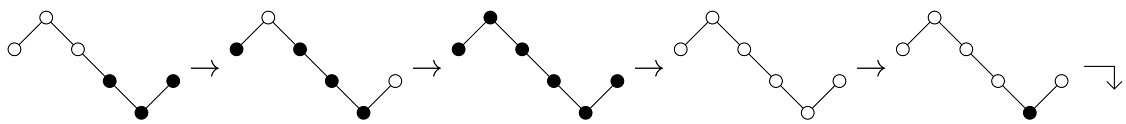
This is a formalization of what we witnessed in Example 1.1.1. Apart from rotation of binary strings, homomesy arises in many other settings. One setting rife with homomesy is that of *order ideals of a posets* under the action of *rowmotion* (see definitions in Section 2). Here are the two orbits of rowmotion on order ideals of a certain fence poset, one of length 11 and one of length 5.

Orbit of length 11:

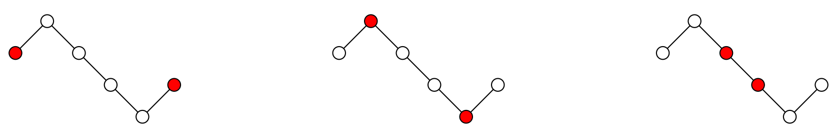




Orbit of length 5:



Label the elements of the fence poset $x_1, \ldots, x_6$ from left to right. We define the characteristic function χ_i to be 1 whenever an order ideal contains x_i and 0 otherwise. The statistics $\chi_1 + \chi_6$, $\chi_2 + \chi_5$, and $\chi_3 + \chi_4$ are each 1-mesic.



$\chi_1 + \chi_6$ $\quad$ $\chi_2 + \chi_5$ $\quad$ $\chi_3 + \chi_4$

For example, the total of number of times x_2 is filed in plus the number of times x_5 is filled in across an orbit is counted by $\chi_2 + \chi_5$. The total of this is the same as the length of the orbit. Homomesies are not always straightforward to prove and one useful technique is a bijection between the two sets that also sends one action to another. Such a bijection is called an *equivariant bijection*. Section 2 and Section 4 contain examples demonstrating the importance of finding the right equivariant bijection.

1.2 Independent sets of a path graph

Let $\mathcal{P}_n$ be the path graph on n vertices. That is $\mathcal{P}$ is the graph with vertices $\{1, \ldots, n\}$ and edges $\{i, i+1\}$ for all $1 \leq i < n$.

Definition 1.2.1. An *independent set* of a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that if $x \in S$ and $y \in S$, then $\{x, y\} \notin E$.

Let $\mathcal{I}(G)$ be the set of all independent sets of G, and for this section we set $\mathcal{I}_n$ to be the set of independent sets of $\mathcal{P}_n$.

Example 1.2.2. For $n = 7$, ●—○—●—○—○—●—○ $\in \mathcal{I}(\mathcal{P}_7)$.

We can represent an independent set $S \in I_n$ of a path graph by a binary string of length n where the ith position is 1 if and only if $i \in S$. For Example 1.2.2 the bitstring would be 1010010. The independence condition means that we get exactly the set of length n binary strings without adjacent substring 11. Next, we introduce an action on independent sets.

Definition 1.2.3 ([19])**.** Define the *toggle* at vertex i, $\tau_i : \mathcal{I}_n \to \mathcal{I}_n$, for $1 \leq i \leq n$ as follows:

$$\tau_i(S) = \begin{cases} S \cup \{i\} & \text{if } i \notin S \text{ and } S \cup \{i\} \in \mathcal{I}_n \\ S & \text{if } i \notin S \text{ and } S \cup \{i\} \notin \mathcal{I}_n \\ S \setminus \{i\} & \text{if } i \in S. \end{cases}$$

It is not hard to see that τ_i is an involution. (If τ_i has no effect, or if an element can be taken out, then it can be put back in and vise versa, hence an involution.) Let 01000 be the binary string representation of an independent set in $\mathcal{I}_5$. If we act on the independent set with τ_4, then we would add 4, since neither 3 or 5 are in the independent set, resulting in 01010. Instead if we act with τ_2, then we will remove 2 from the independent set, resulting in 00000. Finally, if instead we act with τ_3, then nothing will happen because adding 3 would not be an independent set, so the result is the unchanged bitstring 01000.

An important feature of the toggle in this setting is that two toggles commute whenever the nodes they toggle are nonadjacent.

Proposition 1.2.4. *If $G = (V, E)$, $S \in \mathcal{I}(G)$, and $uv \notin E$, then the toggles τ_v and τ_u commute, that is $\tau_v \tau_u(S) = \tau_u \tau_v(S)$.*

Proof. Since u and v are not adjacent, whether u is in an independent set S has no effect on whether v is in S and vice-versa. Hence applying τ_u and τ_v to S in either order gives the same result. $\square$

This is a generalization of [19, Prop 2.2]. We will define $\varphi = \tau_n \tau_{n-1} \ldots \tau_1$ to be the action that toggles the vertices of an independent set of a path graph from left to right.

Definition 1.2.5. Let $\mathcal{S}$ be a set of strings, words, or vectors, of length n and $\tau : \mathcal{S} \to \mathcal{S}$ an invertible map. Let $\mathcal{O}$ be a finite orbit of τ, and let $s = (s_1, \ldots, s_n) \in \mathcal{O}$. An *orbit board* $\mathcal{B}$ is a table of width n and infinite length such that $\mathcal{B}(i, j) = (\tau^i s)_j$. The rows of the table are the elements of $\mathcal{O}$ with the element τs being directly underneath s. Even though the boards are infinite in length, usually just one orbit (each row written once) is written out and is referred to as the orbit board. A *super orbit board* is an orbit board with more than one orbit written out.

Example 1.2.6. Here is the orbit board of φ generated by 01000101. Underneath is the sum of each column.

$$\begin{array}{cccccccc}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
\mathbf{5} & \mathbf{3} & \mathbf{4} & \mathbf{3} & \mathbf{3} & \mathbf{4} & \mathbf{3} & \mathbf{5}
\end{array}$$

We define the statistics $\chi_i : S \to \mathbb{Q}$ to be 1 if $i \in S$ and 0 otherwise. With the action φ and statistic χ we get the following homomesy.

Theorem 1.2.7 ([19])**.** *For $n \geq 2$, under the action of φ on the independent sets of $\mathcal{P}_n$, both the statistics $2\chi_1 + \chi_2$ and $\chi_{n-1} + 2\chi_n$ is 1-mesic.*

The proof for Theorem 1.2.7 is short enough to state here. Suppose you have an independent set that starts with $00\ldots$. After one iteration of ϕ the result will always be $10\ldots$. After the next iteration of ϕ there are two cases, either the string is $101\ldots$ and the next step is $000\ldots$ or the string is $100\ldots$ and the next step is $010\ldots$. In the first case we are already back in case $00\ldots$. The length of this case is 2, explicitly $00\cdots \to 10\cdots \to 00\ldots$ and $2\chi_1 + \chi_2$ will have average 1 over this case. Similarly, the second case's next iteration of ϕ will result in $00\ldots$. So the second case will have length 3, explicitly $00\cdots \to 10\cdots \to 01\cdots \to 00\ldots$, and $2\chi_1 + \chi_2$ with an average of 1 for this case as well. Since every board can be partitioned into these cases and the statistic $2\chi_1 + \chi_2$ has an average of 1 for both types of cases the theorem follows. A similar argument is used for the second homomesy.

A more impressive homomesy can be also be found with an equivariant bijection.

Theorem 1.2.8 ([19]). *Under the action of φ on the independent sets of $\mathcal{P}_n$, the statistic $\chi_i - \chi_{n-i+1}$ is 0-mesic.*

Example 1.2.9. Here is an example of this homomesy in the case where $n = 5$. Notice that the column sums form a palindrome.

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ \hline \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \end{array} \qquad \begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ \hline \mathbf{3} & \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{3} \end{array} \qquad \begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ \hline \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{array}$$

In Example 1.2.6 we get $(\chi_i) = (5, 3, 4, 3, 3, 4, 3, 5)$ which is another example. Theorem 1.2.8 is proven by establishing an equivariant bijection between the binary strings of τ and cyclic shift on compositions of n with each component being 1 or 2. This bijection requires the following lemma.

Lemma 1.2.10 ([19]). *Given an orbit, $\mathcal{O}$, of τ on $\mathcal{P}_n$ fix an independent set $S \in \mathcal{O}$. Let $\chi(i, j) = \chi_j(\tau^i S)$. Suppose $\chi(i, j) = 1$ for $j < n - 1$. The following holds:*

- *If $\chi(i + 2, j) = 1$, then $\chi(i + 1, j + 1) = 0$,*
- *otherwise $\chi(i + 1, j + 1) = 1$.*

Similarly if $j = n - 1$, then $\chi(i + 1, n) = 1$.

Example 1.2.11. Here is an independent set S written above $\varphi(S)$. In red, we can see that since there is no 1 in the fourth position of S, we add a 1 in the third position when applying φ. On the other hand, in blue, we can see that since there is a 1 in the eighth position of S, a 1 will not be placed in the seventh position in $\varphi(S)$.

$$\begin{array}{rcccccccc} S = & 0 & \boxed{1} & 0 & 0 & 0 & \boxed{1} & 0 & \boxed{1} \\ \varphi(S) = & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 \end{array}$$

This is a lemma best proven by inspection. View an orbit board as a matrix of bits where each row is an independent set. When applying τ to an independent set, if the jth entry is 1 for $j < n-1$, then the jth bit is changed to a 1 and the $(j+1)$st bit will be changed to a 1 if and only if the $(j+2)$nd is 0. Using this lemma, the 1's of an orbit board can now be partitioned into *jump snakes*. We can write a jump snake as a composition of $n-1$ with 1's and 2's, where diagonal moves are marked with 1 and horizontal moves are marked with 2 so the sum of the composition is always $n-1$.

Example 1.2.12. The jump snake partition for Example 1.2.6.

$$\begin{array}{cccccccc}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}$$

Here are the compositions of 7 corresponding to the corresponding jump snakes: 11122 (for the snake starting the in the bottom left corner), 11221, 12211, 22111, and 21112.

Theorem 1.2.13 ([19]). *In an orbit board $\mathcal{B}$, consider a snake starting on the ith line. Let c be the snake's composition. Consider the least i', $i' > i$, for which the first entry of the i'th line is* 1. *(This is where the "next" snake begins.)*

1. *If c starts with* 1*, then $i' = i + 3$.*
2. *If c starts with* 2*, then $i' = i + 2$.*
3. *The composition for the snake starting on the $S^{i'}$ line is the left cyclic rotation of c.*

Using Theorem 1.2.13(3), we can prove the palindromic homomesy (Theorem 1.2.8) using the following bijection between the 1's in the orbit board. A partial sum of $i - 1$ in the composition means there is a 1 is the ith position in some orbit element S. Suppose this partial sum is of length k, then after k iterations of leftward cyclic shift the partial sum will have moved to the right side of the composition. This indicates a 1 in the $n - i + 1$st position of some other element in the orbit board. Full details are available in [19, Thm 2.25].

Using this setting as a launching point, there are a few different directions in which one can generalize these results. There is an equivariant bijection from this toggle product τ on independent sets to rowmotion on order ideals of a certain zigzag poset. We will see this later with recent results in rowmotion on fence posets in Section 2. The notion of independent set can be generalized to m-independent set where each pair of 1's is separated by at least m 0's. These m-independent sets come with a generalization of the rotation action which we will see in Section 3.3.1. Finally we will see in Section 5 another generalization to colored independent sets which we call *partial proper colorings.*

Chapter 2

Rowmotion

2.1 Definitions

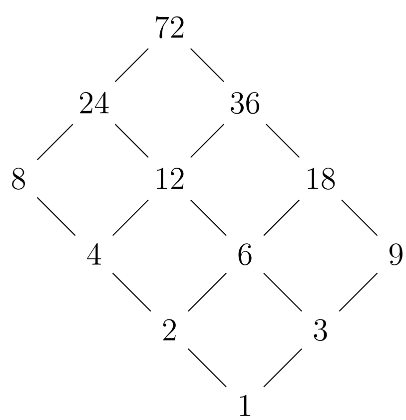
Rowmotion (Definition 2.1.5) is an invertible action on order ideals (or analogously antichains) of a partially ordered set (poset) (Definition 2.1.1) which has received much interest within dynamical algebraic combinatorics. Rowmotion was introduced by Duchet in [9] and first studied for the Boolean lattice and the product of two chains by Brouwer and Schrijver [5, 4]. Rowmotion and its generalizations have been investigated by many authors, for example [6, 27, 33, 34]. See, in particular, the survey articles of Roby [28] and Striker [32] and the references therein. We begin this section with a brisk overview of Posets with notation mostly borrowed from *Enumerative Combinatorics, Vol. 1* [31].

Definition 2.1.1 ([31])**.** A *partially ordered set* P (or poset) is a set with a binary relation $\leq_P$ (or just $\leq$ when unambiguous) which is reflexive, antisymetric, and transitive.

We will often write our poset P with relation $\leq_P$ as the pair $(P, \leq_P)$. We will also abuse notation and drop the P to write $x < y$ when $x, y \in P$ are distinct and there is no ambiguity. Within a poset one element *covers* a smaller element if there is no element that sits in-between, that is, y covers x if there does not exist $z \in P$ such that $x < z < y$. When y covers x we may write $x \lessdot y$ or $y \gtrdot x$. When there is no relation between x and y we say x and y are *incomparable*,

There are many common examples of posets of infinite size such as $\mathbb{N}$ with its natural ordering but we will focus on finite posets. It is often useful to represent a poset by its *Hasse Diagram*, a graph whose vertices are the elements of P and whose edges represent cover relations.

Example 2.1.2. The divisors of 72 form a poset with relation $x \leq y$ if and only if x divides y.



Definition 2.1.3 ([31])**.** Let $(P, \leq)$ be a poset.

1. An *order ideal* of P is a subset I of P such that if $y \in I$ and $x \leq y$, then $x \in I$. Similarly, an *order filter* of P is a subset F of P such that if $y \in F$ and $y \leq x$, then $x \in F$.

2. A *chain* is a subset C of P such that if $x, y \in C$, then either $x \leq y$ or $y \leq x$. An *antichain* of P is a subset A of P such that if $x, y \in A$, then x and y are incomparable.

Let $J(P)$ and $A(P)$ denote the set of all order ideals and antichains of a poset P respectively.

Example 2.1.4. Using the poset in Example 2.1.2, we see $\{1,2,3\}$ is an order ideal, $\{9,12,18,24,36,72\}$ is an order filter, $\{2,18,36,72\}$ is a chain, and $\{6,8,9\}$ is an antichain.

Another useful poset construction is the Cartesian product of posets. For posets P and Q we define the poset $(P \times Q, \leq_{P\times Q})$ (usually just written $P \times Q$) where $(p,q) \leq_{P\times Q} (p',q')$ if and only if $p \leq_P p'$ and $q \leq_Q q'$.

Let $\mathcal{L}(P)$ be the set of injective function $f : P \to [n]$ such that $f(x) < f(y)$ if and only if $x <_P y$. The set $\mathcal{L}(P)$ is called the set of *linear extensions* or *total orderings* of P.

Definition 2.1.5 ([20])**.** Let P be a generic poset. We define natural bijections between the sets $\mathcal{J}(P)$ of all order ideals of P, $\mathcal{F}(P)$ of all order filters of P, and $\mathcal{A}(P)$ of all antichains of P.

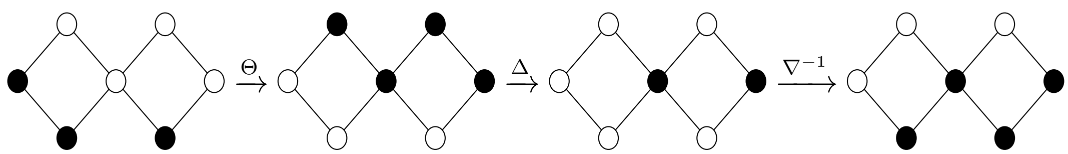
- The map $\Theta : 2^P \to 2^P$ where $\Theta(S) = P \backslash S$ is the *complement* of S (sending order ideals to filters and vice versa).

- The *up-transfer* $\nabla : \mathcal{J}(P) \to \mathcal{A}(P)$, where $\nabla(I)$ is the set of maximal elements of I. For an antichain $A \in \mathcal{A}(P)$, $\nabla^{-1}(A) = \{x \in P : x \leq y \text{ for some } y \in A\}$.

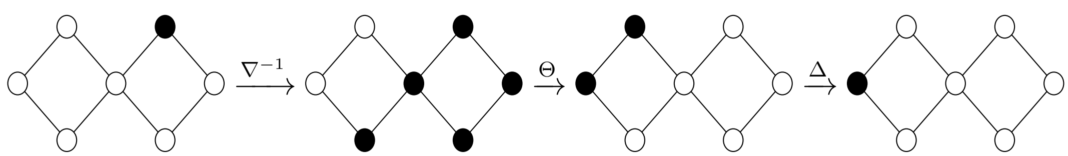
- The *down-transfer* $\Delta : \mathcal{F}(P) \to \mathcal{A}(P)$, where $\Delta(F)$ is the set of minimal elements of F. For an antichain $A \in \mathcal{A}(P)$, $\Delta^{-1}(A) = \{x \in P : x \geq y \text{ for some } y \in A\}$.

Order ideal rowmotion is the map $\rho_J : \mathcal{J}(P) \to \mathcal{J}(P)$ given by $\rho_J = \nabla^{-1} \circ \Delta \circ \Theta$. *Antichain rowmotion* is the map $\rho_A : \mathcal{A}(P) \to \mathcal{A}(P)$ given by $\rho_A = \Delta \circ \Theta \circ \nabla^{-1}$.

Example 2.1.6. Here is an example of one iteration of ρ_J on an order ideal with the action broken down into its three steps.



Example 2.1.7. Here is an example of one iteration of ρ_A on an antichain with the action broken down into its three steps.



One could also define rowmotion on order filters, which gives the most natural lifting to PL-rowmotion on chain polytopes [11].

2.2 Rowmotion as a product of toggles

Rowmotion has an alternate definition as a composition of toggling involutions, which has proven useful for understanding and generalizing many of its properties. Cameron and Fon-der-Flaass [6] showed that for any finite poset P, rowmotion can be realized as "toggling once at each element of P along any linear extension (from top to bottom)". Other toggling orders also lead to interesting maps, such as Striker–Williams "promotion" (of order ideals) of a poset, which is toggling from left-to-right along "files" of a poset [33].

Definition 2.2.1. For each fixed $i \in P$ define the *antichain toggle* $\tau_i : \mathcal{A}(P) \to \mathcal{A}(P)$ by

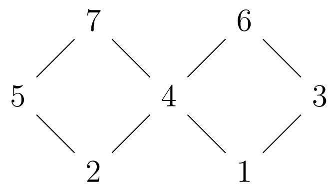
$$\tau_i(A) = \begin{cases} A \smallsetminus \{i\} & \text{if } i \in A \\ A \cup \{i\} & \text{if } i \notin A \text{ and } A \cup \{i\} \in \mathcal{A}(P) \\ A & \text{if } i \notin A \text{ and } A \cup \{i\} \notin \mathcal{A}(P) \end{cases}$$

Similarly, we defined the *order-ideal toggle* $\widehat{\tau}_i : \mathcal{J}(P) \to \mathcal{J}(P)$ by

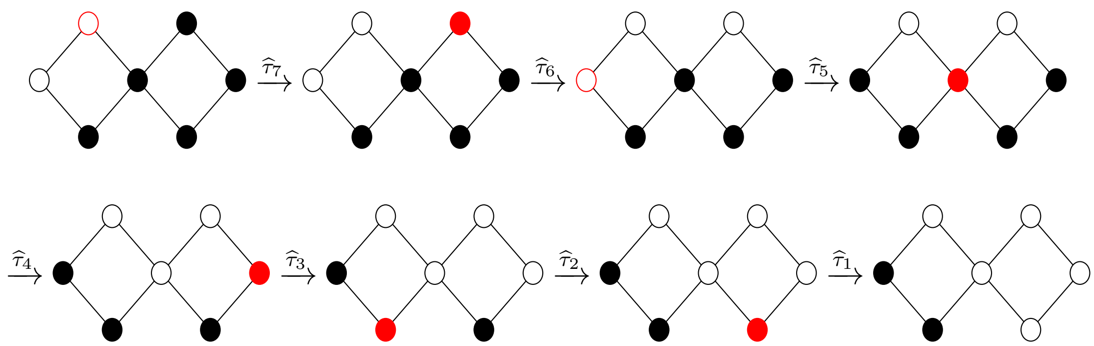
$$\widehat{\tau}_i(I) = \begin{cases} I \smallsetminus \{i\} & \text{if } i \in I \text{ and } I \smallsetminus \{i\} \in \mathcal{J}(P) \\ I \cup \{i\} & \text{if } i \notin I \text{ and } I \cup \{i\} \in \mathcal{J}(P) \\ I & \text{otherwise.} \end{cases}$$

It is an easy exercise to show that both antichain toggles [6, §2] and order ideal toggles [19, §5] are involutions. Note that one can always toggle an element out of an antichain to get a different antichain, which explains a small asymmetry in the definitions as stated.

Example 2.2.2. We will toggle each node down a linear extension, at each step we consider whether or not to toggle the red node in or out. Fix the following linear extension



For the following linear extension we toggle the elements of this poset from left-to-right and top-to-bottom.



Proposition 2.2.3 ([6])**.** *Let $x_1, x_2, \ldots, x_n$ be any linear extension (i.e., any order-preserving listing of the elements) of a finite poset P with n elements. Then the composite map $\widehat{\tau}_{x_1}\widehat{\tau}_{x_2}\cdots\widehat{\tau}_{x_n}$ coincides with the rowmotion operation $\widehat{\rho}_J$.*

Rowmotion on order ideals or antichains of a poset has been one focus of research in dynamical algebraic combinatorics with the goal of finding homomesies. One area of recent development was on the family of posets known as fence posets.

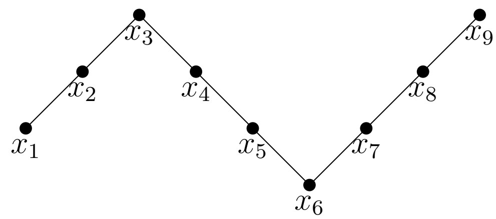
2.3 Rowmotion on fence posets

A *fence* poset, $F = \{x_1, \ldots, x_n\}$, is a poset with elements

$$x_1 \lessdot x_2 \lessdot \cdots \lessdot x_j \gtrdot x_{j+1} \gtrdot \cdots \gtrdot x_k \lessdot \ldots$$

The maximal chains of F are called *segments*. Elements that are in more than one segment are called *shared* elements, and all other elements are *unshared* elements.

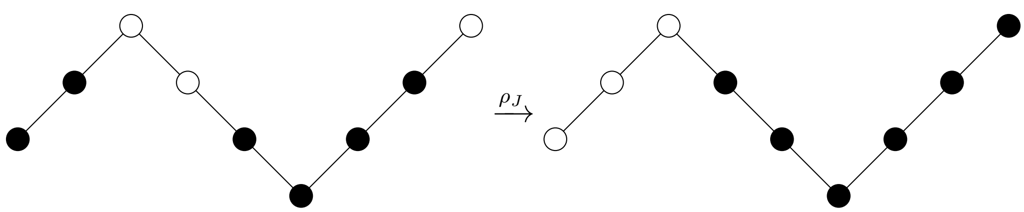
Example 2.3.1. Here is a fence with 9 elements.



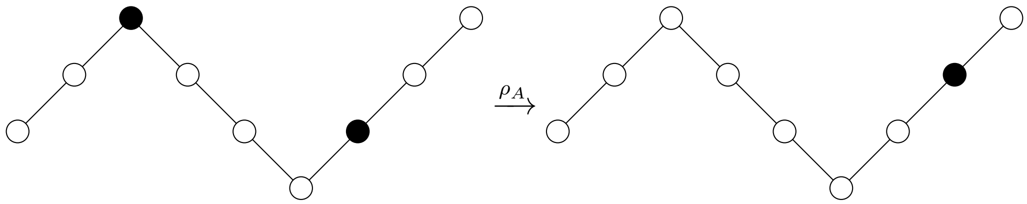
This fence poset has three segments, and the elements x_3 and x_6 are shared.

We will denote each F by the number of unshared elements in each segment. That is, we will write $F(\alpha)$ where $\alpha = (\alpha_1, \ldots, \alpha_m)$ to mean the fence poset with m segments and α_j unshared elements in the jth segment. So the example above is $F(2, 2, 3)$. Fences have important connections with cluster algebras [7, 24, 26, 29, 30, 35], q-analogues [22], unimodality [7, 14, 16, 21, 23], and Young diagrams [25]. The study of rowmotion on fence posets was initiated in [13].

Example 2.3.2. Here is an example of order-ideal rowmotion on a small poset.



Here is another example but with antichain rowmotion.

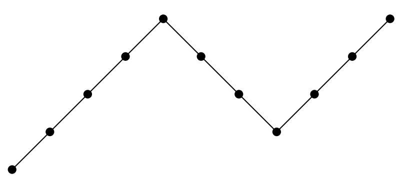
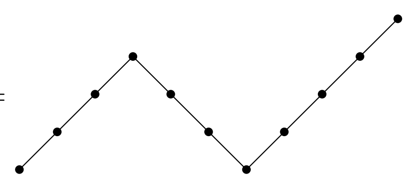


In this section we will investigate both order-ideal rowmotion and antichain rowmotion. To help distinguish between the two settings, we will denote order-ideal rowmotion with ρ_J and similarly the cardinality statistic on an order ideal will be denoted by $\widehat{\chi}$.

2.3.1 Dual posets

For any poset P we can define the dual of P, denoted P^*, to be the poset which reverses the ordering on P (on the same set of elements).

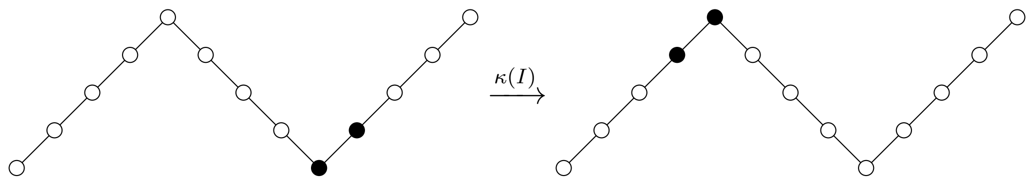
Example 2.3.3. Here we have $P = F(4, 2, 3)$ and $P^* = F(3, 2, 4)$.

$P =$  $P^* =$ 

Call P *self-dual* if $P \cong P^*$. When P is self-dual we can say something about homomesies for the cardinality statistic with order-ideal rowmotion. We'll take a brief aside into dual posets to state this proposition formally.

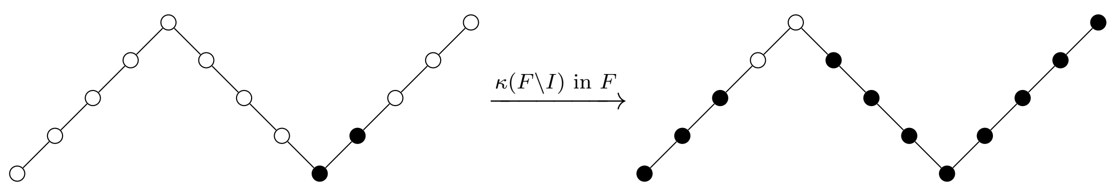
Definition 2.3.4. Let P be self-dual and define $\kappa : P \to P^*$ to be the map that sends $x \in P$ to $y \in P^*$ found by flipping the Hasse diagram upside-down. In particular $\kappa(I) \in \mathcal{F}(P)$ for all $I \in \mathcal{J}(P)$.

Example 2.3.5. The poset $P = F(4,3,4)$ has an order ideal I on the left and the order filter $\kappa(I)$ on the right.



Definition 2.3.6. Let P be self-dual. Define the complement $\overline{I}$ of I to be the order ideal obtained by flipping $P\backslash I$ upside down. That is, $\overline{I} = \kappa(P\backslash I)$ as an order ideal in P.

Example 2.3.7. Let $F = F(4,3,4)$.



Lemma 2.3.8. *Let P be a self-dual poset, $P = P^*$. Then*

$$\rho_J^{-1}(\overline{I}) = \overline{\rho_J(I)}$$

for all $I \in \mathcal{J}(P)$.

Proof. We can write rowmotion as the composition of three maps, $\rho_J = \nabla^{-1} \circ \Delta \circ \Theta$, and the complement map as the composition $\Theta \circ \kappa$, or identically as $\kappa \circ \Theta$. It suffices to see that the following diagram commutes for any $I_1 \in \mathcal{J}(P)$.

$$\begin{tikzcd}
I_1 \arrow[r, "\Theta"] & F_1 \arrow[r, "\Delta"] \arrow[d, "\kappa"] & A_1 \arrow[r, "\nabla^{-1}"] \arrow[d, "\kappa"] & I_2 \arrow[d, "\kappa"] & \\
& \overline{I_1} \arrow[r, "\nabla"] & A \arrow[r, "\Delta^{-1}"] & F \arrow[r, "\Theta"] & I
\end{tikzcd}$$

The first square commutes because taking the minimal elements of an order filter when flipped upside-down will be the maximal elements of F flipped upside-down.

The second square commutes because the downward saturation of an antichain corresponds to the upwards saturation of $\kappa(A)$ in P^*.

With this we get the following from the diagram. Taking the higher path we obtain,

$$I = \Theta \circ \kappa \circ (\nabla^{-1} \circ \Delta \circ \Theta(I_1)) = \overline{\rho_J(I_1)},$$

and taking the lower path we obtain,

$$I = \Theta \circ \Delta^{-1} \circ \nabla \circ (\kappa \circ \Theta(I_1)) = \rho_J^{-1}(\overline{I_1}).$$

$\square$

The following proposition gives us a result for homomesies whenever P is self-dual and every order-ideal rowmotion orbit of P is self-dual, in the sense that, if $I \in \mathcal{J}(P)$ generates an orbit $\mathcal{O}$ and $\overline{I}$ generates an orbit $\overline{\mathcal{O}}$, then $\mathcal{O} = \overline{\mathcal{O}}$.

Proposition 2.3.9. *Let P be self-dual with $n = \#P$, and fix an order-reversing bijection $\kappa : P \to P$. Let $I \in \mathcal{J}(P)$. If $I \in \mathcal{O}$ and $\overline{\mathcal{O}}$ is the orbit generated by $\overline{I}$, then $\#\mathcal{O} = \#\overline{\mathcal{O}}$ and*

$$\frac{\widehat{\chi}(\mathcal{O}) + \widehat{\chi}(\overline{\mathcal{O}})}{\#\mathcal{O} + \#\overline{\mathcal{O}}} = \frac{n}{2}.$$

Furthermore, if $\mathcal{O} = \overline{\mathcal{O}}$ for all $I \in \mathcal{I}(P)$, then $\widehat{\chi}$ is $n/2$-mesic.

Proof. Fix $I \in \mathcal{O}$. From Lemma 2.3.8 we know $\overline{\rho_J^k(I)} = \rho_J^{-k}(\overline{I})$. If $\#\mathcal{O} = m$, then $\rho_J^m(I) = I$ and it follows $\rho_J^m(\overline{I}) = \overline{\rho_J^m(I)} = \overline{I}$, so $\#\overline{\mathcal{O}} = m$. Now the sum $\mathcal{O} \oplus \overline{\mathcal{O}}$ can be partitioned into pairs $\{I, \overline{I}\}$ and singleton $\{I\}$ with $I = \overline{I}$. In each pair and singleton, the average of $\widehat{\chi}$ is $n/2$, so the same is true when considered over the whole orbit. When $\mathcal{O} = \overline{\mathcal{O}}$, the result reduces to the homomesy. $\square$

A fence poset $F(\alpha)$ is self dual whenever α is a palindrome of odd length. It turns out that the study of independent sets of a path graph (Subsection 1.2) is subsumed by the study of fence posets, because there is an equivariant bijection from independent sets of a path graph to order ideals of the fence poset $F = F(1, 0, 0, \ldots, 0, 1)$ called a *zigzag* poset. The equivariant bijection takes the toggling from left-to-right map, φ, to rowmotion on order ideals, ρ_J. [19].

2.4 Toggling rowmotion on fence posests

In this section we prove that the order in which the vertices of a fence poset are order-ideal toggled does not change the set of homomesies (Theorem 2.4.5). To accomplish this we define the *toggle groups* for antichains and order ideals, then utilize a strategy similar to one Joseph and Roby used in a related context [19, §2.3], which in turn uses ideas from Einstein, et al. [10, §2–3]. We state an analogous result for antichain toggles (Conjecture 2.4.10), which has thus far evaded our current proof techniques.

Proposition 2.4.1 ([6]). *Let P be a finite poset.*

1. *For every $x \in P$, $\widehat{\tau}_x$ is an involution, i.e., $\widehat{\tau}_x^2 = 1$.*

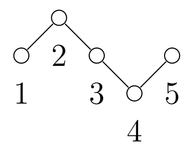


FIGURE 2.1: $F(1,1,1)$

2. *For every $x, y \in P$ where neither x covers y nor y covers x, the toggles commute, i.e., $\widehat{\tau}_x \widehat{\tau}_y = \widehat{\tau}_y \widehat{\tau}_x$.*

Let $\mathcal{T}_P$ be the *antichain toggle group* of $\mathcal{A}(P)$, that is, $\mathcal{T}_P$ is the subgroup generated by the antichain toggles within $\mathfrak{S}_{A(P)}$. Similarly, we define $\widehat{\mathcal{T}}_P$ to be the *order-ideal toggle group* of $\mathcal{J}(P)$, that is, $\mathcal{T}_P$ is the subgroup generated by the antichain toggles within $\mathfrak{S}_{J(P)}$. For a finite poset whose nodes are labeled by $[n]$, we call an element w in $\mathcal{T}_P$ (resp. $\widehat{\mathcal{T}}_P$) a *Coxeter element* if it is a product (in some order) of $\tau_1, \tau_2, \dots, \tau_n$ (resp. $\widehat{\tau}_1, \widehat{\tau}_2, \dots, \widehat{\tau}_n$), each used exactly once.

Example 2.4.2. Consider the fence poset $F = F(1,2,1)$ with vertices labeled 1, 2, 3, 4, and 5 from left to right, as in Figure 2.1. Two particular elements of $\mathcal{T}_F$ would be $\mathrm{Pro} := \tau_5\tau_4\tau_3\tau_2\tau_1$ and $\rho_J = \tau_2\tau_5\tau_3\tau_1\tau_4$, promotion and rowmotion respectively.

Proposition 2.4.3 ([19])**.** *Within the antichain toggle group $\mathcal{T}_F$ of any fence F, the order of the map $\tau_i \circ \tau_j$ is*

$$\begin{cases} 1 & \textit{if } i = j, \\ 2 & \textit{if } i \textit{ and } j \textit{ are incomparable}, \\ 2, 3, \textit{or } 6 & \textit{if } i < j \textit{ or } j < i. \end{cases}$$

Proof. The first two cases are straightforward. Assume $i < j$. To show the order of

$\tau_j\tau_i$ is 6 we will show there is an orbit of size 2, an orbit of size 3, and none larger. Notice that any orbit of $\tau_j\tau_i$ contains at most $A, A\cup\{i\}, A\cup\{j\}$, for any $A \in \mathcal{A}(P)$. Particularly, $\{\emptyset, i, j\}$ is an orbit of order 3. However, it could be that j is a maximal element and there is $k \in A$ such that k and i are incomparable but $j > k$. In this case the order of $\tau_j\tau_i$ is 2, generating the orbit $(A, A\cup\{i\})$. $\square$

An analogous statement holds for order-ideal toggles.

Proposition 2.4.4. *Within the order-ideal toggle group $\widehat{\mathcal{T}}_F$ of any fence F, the order of the map $\widehat{\tau}_i \circ \widehat{\tau}_j$ is*

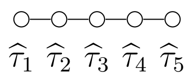
$$\begin{cases} 1 & \textit{if } i = j, \\ 2 & \textit{if } |i-j| \geq 2 \\ 2, 3, \textit{or } 6 & \textit{if } |i-j| = 1. \end{cases}$$

So $\mathcal{T}_P$ and $\widehat{\mathcal{T}_P}$ are each quotients of Coxeter groups. (The Coxeter groups given by just the relations in Prop. 2.4.3 or in Prop. 2.4.4 are infinite, while $\mathcal{T}_P \subset \mathfrak{S}_{A(P)}$ and $\widehat{\mathcal{T}_P} \subset \mathfrak{S}_{J(P)}$ are finite.) Within each group, a given product of toggles may give the same result when the toggles are rearranged in certain ways (but not in others), since some toggles commute with each other. The next proposition shows that even if rearranging the order of factors of a Coxeter element results in a different map, then the two maps will give the same homomesies, at least for those writable as linear combinations of poset-element characteristic functions.

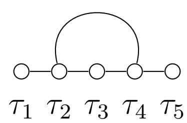
Theorem 2.4.5. *Let F be a fence poset, and let w and w' be two Coxeter elements in the order-ideal toggle group $\widehat{\mathcal{T}}_F$. Then any statistic which is a linear combination of indicator functions $\widehat{\chi}_j$ is c-mesic under the action of w if and only if it is c-mesic under the action of w'.*

Definition 2.4.6. For any toggle group $\mathcal{T}$ we define the *base graph* $\Gamma_{\mathcal{T}}$ whose vertex set is the generators with an edge connecting two vertices if and only if the associated elements do not commute.

Example 2.4.7. Let $F = F(1, 2, 1)$ as in Figure 2.1. Following Proposition 2.4.4, the base graph of $\widehat{\mathcal{T}}_F$ is the path graph of length 5.



Similarly, following Proposition 2.4.3 the base graph of $\mathcal{T}_F$ is



An old result of Bourbaki [3, 2, V, §6, n$^{\text{deg}}$, Lemma 1] states that if the base graph has no cycles, then $\widehat{\tau}_{\nu(1)} \cdots \widehat{\tau}_{\nu(n)}$ is conjugate to $\widehat{\tau}_1 \cdots \widehat{\tau}_n$ for any permutation $\nu \in \mathfrak{S}_n$. (See also Lemma 5.1 of [33].) This is the situation within the order-ideal toggle group $\widehat{\mathcal{T}}_F$ (since the edges are the covering relations), but not the antichain toggle group $\mathcal{T}_F$ (in which any two comparable elements are connected). In order to prove the homomesies, we need a more explicit construction that shows we can get from a Coxeter element to any other via a sequence of *admissible conjugations*, which we now describe.

We associate with each Coxeter word an acyclic orientation of the base graph, defined as follows: We direct the edge connecting i and $i + 1$ in the direction of i (resp. $i + 1$) if $\widehat{\tau}_i$ appears to the right (resp. left) of $\widehat{\tau}_{i+1}$ in the word. For example, the Coxeter element $\widehat{\tau}_1\widehat{\tau}_5\widehat{\tau}_2\widehat{\tau}_4\widehat{\tau}_3$ is associated with the following orientation of the base graph of $F(1,2,1)$.

$$\begin{array}{ccccccccc} \circ & \to & \circ & \to & \circ & \leftarrow & \circ & \leftarrow & \circ \\ \widehat{\tau}_1 & & \widehat{\tau}_2 & & \widehat{\tau}_3 & & \widehat{\tau}_4 & & \widehat{\tau}_5 \end{array}$$

Notice that in any Coxeter element each toggle at a source vertex appears to the left of any other toggle with which it does not commute. We call such toggles *initial* in the Coxeter word. Similarly any toggle whose noncommuting toggles are all to the right in the Coxeter are called *final* and are associated with sinks in the base graph orientation. Since toggles are involutions, conjugating by an initial or final toggles gives another Coxeter element. For example, conjugating $w = \widehat{\tau}_2\widehat{\tau}_5\widehat{\tau}_4\widehat{\tau}_3\widehat{\tau}_1$ by $\widehat{\tau}_3$ gives

$$\widehat{\tau}_3(\widehat{\tau}_2\widehat{\tau}_5\widehat{\tau}_4\widehat{\tau}_3\widehat{\tau}_1)\widehat{\tau}_3 = \widehat{\tau}_3(\widehat{\tau}_2\widehat{\tau}_5\widehat{\tau}_4\widehat{\tau}_1\widehat{\tau}_3)\widehat{\tau}_3 = \widehat{\tau}_3\widehat{\tau}_2\widehat{\tau}_5\widehat{\tau}_4\widehat{\tau}_1$$

Notice that this conjugated $\widehat{\tau}_3$ from the right to the left, thus changing the vertex from a sink to a source. These conjugations by initial or final toggles are called *admissible*. Conjugations of Coxeter elements which are not admissible do not necessarily result in Coxeter elements. Eriksson and Eriksson [15, Theorem 1.1] show that any two Coxeter elements are conjugate if and only if there is a sequence of admissible conjugations. (In their language changing sources to sinks and vice-versa in the base graph is called “chip-firing”, which results in a “rotation”, which we would call an admissible conjugation.)

Example 2.4.8. Here is an example of conjugating $w = \widehat{\tau}_1\widehat{\tau}_5\widehat{\tau}_2\widehat{\tau}_4\widehat{\tau}_3$ by a sequence of admissible conjugations to arrive at $w' = \widehat{\tau}_1\widehat{\tau}_2\widehat{\tau}_3\widehat{\tau}_4\widehat{\tau}_5$. Note that the arrows of the base graph do not change when moving commuting elements past one another.

$$\widehat{\tau}_1\widehat{\tau}_5\widehat{\tau}_2\widehat{\tau}_4\widehat{\tau}_3 \qquad \underset{\widehat{\tau}_1}{\circ}\to\underset{\widehat{\tau}_2}{\circ}\to\underset{\widehat{\tau}_3}{\circ}\leftarrow\underset{\widehat{\tau}_4}{\circ}\leftarrow\underset{\widehat{\tau}_5}{\circ}$$

conjugate by $\widehat{\tau}_3$

$$\widehat{\tau}_3\widehat{\tau}_1\widehat{\tau}_5\widehat{\tau}_2\widehat{\tau}_4 \qquad \underset{\widehat{\tau}_1}{\circ}\to\underset{\widehat{\tau}_2}{\circ}\leftarrow\underset{\widehat{\tau}_3}{\circ}\to\underset{\widehat{\tau}_4}{\circ}\leftarrow\underset{\widehat{\tau}_5}{\circ}$$

move $\widehat{\tau}_2$ to the right

$$\widehat{\tau}_3\widehat{\tau}_1\widehat{\tau}_5\widehat{\tau}_4\widehat{\tau}_2 \qquad \underset{\widehat{\tau}_1}{\circ}\to\underset{\widehat{\tau}_2}{\circ}\leftarrow\underset{\widehat{\tau}_3}{\circ}\to\underset{\widehat{\tau}_4}{\circ}\leftarrow\underset{\widehat{\tau}_5}{\circ}$$

conjugate by $\widehat{\tau}_2$

$$\widehat{\tau}_2\widehat{\tau}_3\widehat{\tau}_1\widehat{\tau}_5\widehat{\tau}_4 \qquad \underset{\widehat{\tau}_1}{\circ}\leftarrow\underset{\widehat{\tau}_2}{\circ}\to\underset{\widehat{\tau}_3}{\circ}\to\underset{\widehat{\tau}_4}{\circ}\leftarrow\underset{\widehat{\tau}_5}{\circ}$$

move $\widehat{\tau}_5$ to the left

$$\widehat{\tau}_5\widehat{\tau}_2\widehat{\tau}_3\widehat{\tau}_1\widehat{\tau}_4 \qquad \underset{\widehat{\tau}_1}{\circ}\leftarrow\underset{\widehat{\tau}_2}{\circ}\to\underset{\widehat{\tau}_3}{\circ}\to\underset{\widehat{\tau}_4}{\circ}\leftarrow\underset{\widehat{\tau}_5}{\circ}$$

conjugate by $\widehat{\tau}_5$

$$\widehat{\tau}_2\widehat{\tau}_3\widehat{\tau}_1\widehat{\tau}_4\widehat{\tau}_5 \qquad \underset{\widehat{\tau}_1}{\circ}\leftarrow\underset{\widehat{\tau}_2}{\circ}\to\underset{\widehat{\tau}_3}{\circ}\to\underset{\widehat{\tau}_4}{\circ}\to\underset{\widehat{\tau}_5}{\circ}$$

move $\widehat{\tau}_1$ to the right

$$\widehat{\tau}_2\widehat{\tau}_3\widehat{\tau}_4\widehat{\tau}_5\widehat{\tau}_1 \qquad \underset{\widehat{\tau}_1}{\circ}\leftarrow\underset{\widehat{\tau}_2}{\circ}\to\underset{\widehat{\tau}_3}{\circ}\to\underset{\widehat{\tau}_4}{\circ}\to\underset{\widehat{\tau}_5}{\circ}$$

conjugate by $\widehat{\tau}_1$

$$\widehat{\tau}_1\widehat{\tau}_2\widehat{\tau}_3\widehat{\tau}_4\widehat{\tau}_5 \qquad \underset{\widehat{\tau}_1}{\circ}\to\underset{\widehat{\tau}_2}{\circ}\to\underset{\widehat{\tau}_3}{\circ}\to\underset{\widehat{\tau}_4}{\circ}\to\underset{\widehat{\tau}_5}{\circ}$$

This leads to the following lemma, from which Theorem 2.4.5 follows immediately.

Lemma 2.4.9. *Fix a fence poset F. Let w and w' be Coxeter elements of $\widehat{\mathcal{T}}_F$ which differ by a single admissible conjugation. Any statistic which is a linear combination of indicator functions $\widehat{\chi}_j$ is c-mesic under the action of w if and only if it is c-mesic under the action of w'.*

Proof. Suppose $w' = \widehat{\tau}_k w \widehat{\tau}_k$ where $\widehat{\tau}_k$ is initial or final. It follows that $w'\widehat{\tau}_k(I) = \widehat{\tau}_k w(I)$

for any $I \in \mathcal{J}(F)$. Let $\mathcal{O}$ be an orbit of w generated by I, that is,

$$\mathcal{O} = (I^0, I^1, \ldots, I^{m-1})$$

where $I^i = w^i(I)$ for any $i \in \mathbb{Z}$ and $I^m = I^0 = I$. Let $I' = \widehat{\tau}_k I$, then $w'I' = \widehat{\tau}_k w \widehat{\tau}_k \widehat{\tau}_k I = \widehat{\tau}_k wI = \widehat{\tau}_k I^1$. This gives us a w' orbit $\mathcal{O}' = (\widehat{\tau}_k I^0, \widehat{\tau}_k I^1, \ldots, \widehat{\tau}_k I^{m-1})$. Furthermore, this induces a bijection between the orbits of w and orbits of w'. Our aim is now to show that homomesies are preserved through this bijection. To accomplish this, it suffices to show that the indicator functions are preserved. If $j \neq k$, then

$$\sum_{I \in \mathcal{O}} \widehat{\chi}_j I = \sum_{I \in \mathcal{O}'} \widehat{\chi}_j I$$

since toggling by $\widehat{\tau}_k$ does not change whether or not j is in I. On the other hand, assume τ_k is final, that is we can move the toggle to the right, so applying τ_k to I^i changes $k \in I$ just as w does and $\widehat{\chi}_k(\tau_k(I^i)) = \widehat{\chi}_k(I^{i+1})$. Otherwise, τ_k is initial, that is, we can move the toggle to the left, so applying τ_k to I^i cancels with the τ_k in w and $\widehat{\chi}_k(\tau_k(I^i)) = \widehat{\chi}_k(I^{i-1})$. Since the orbit is length m, we have $I^{-1} = I^{m-1}$ and $I^m = I^0 = I$. Regardless of whether τ_k is initial or final we obtain,

$$\sum_{I \in \mathcal{O}} \widehat{\chi}_k I = \widehat{\chi}_k(I^0) + \widehat{\chi}_k(I^1) + \cdots + \widehat{\chi}_k(I^{m-1}) = \sum_{I \in \mathcal{O}'} \widehat{\chi}_k I$$

Since the indicator functions are preserved through admissible conjugation, so are the homomesies. □

Theorem 2.4.5 follows since any two Coxeter elements are connected by a sequence of admissible conjugations. The analogous results for antichain toggles appears to be true as well experimentally.

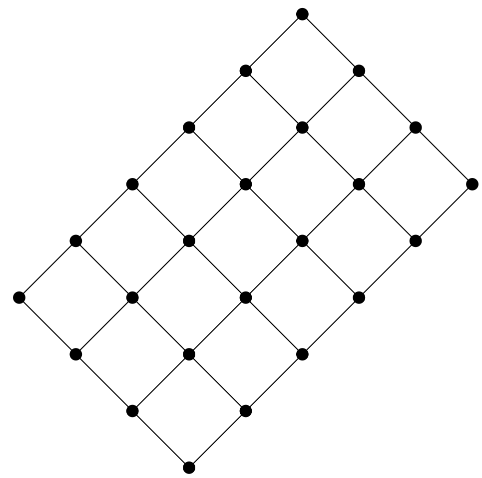
Conjecture 2.4.10. *Let F be a fence poset, and let w and w' be two Coxeter elements in the antichain toggle group $\mathcal{T}_F$. Then any statistic which is a linear combination of indicator functions χ_j is c-mesic under the action of w if and only if it is c-mesic under the action of w'.*

The base graph of the antichain toggle group on the fence poset is not a path in general. Any segment with interior elements will form cycles in the base graph. Eriksson and Eriksson [15, Proposition 4.1] show that admissible conjugations induce an equivalence relation on the set of Coxeter elements. There is only one conjugacy class for the path graph, so admissible conjugations suffice to prove our result for order-ideal toggles. However, there are multiple equivalence classes when the base graph has cycles, so following a similar strategy to that of Proposition 2.4.5 breaks down.

2.5 Rowmotion on a product of two chains

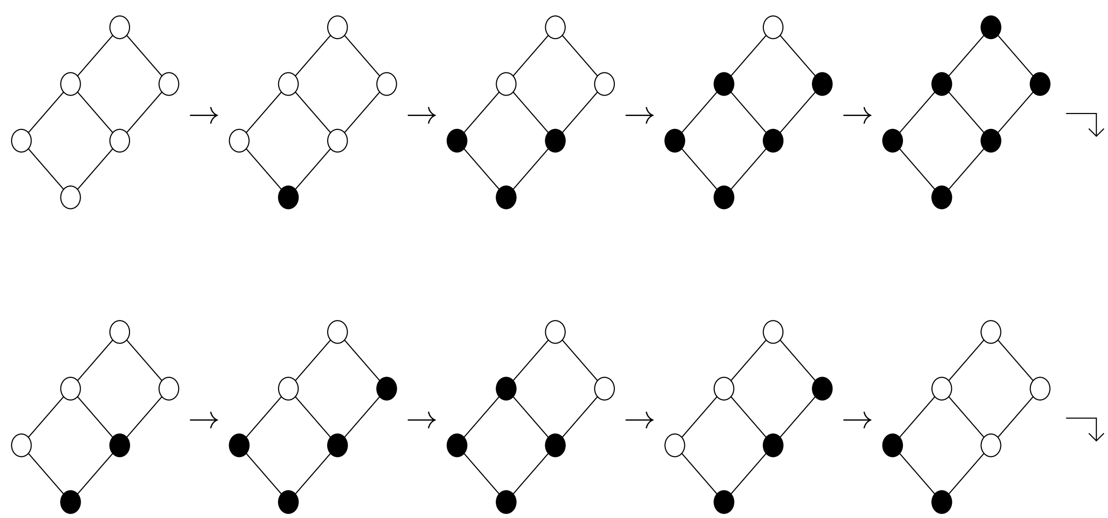
An early example of rowmotion exhibiting homomesy was for order ideals of a poset which is a product of two chains. Let a and b be positive integers and $[a] \times [b]$ the poset obtained by taking the Cartesian product of $[a]$ and $[b]$.

Example 2.5.1. Here is the poset $[4] \times [6]$.



Theorem 2.5.2 ([27], Section 3.3, Theorem 23)**.** *The cardinality statistic is c-mesic under the action of rowmotion ρ_J on $\mathcal{J}([a] \times [b])$, with $c = ab/2$.*

Example 2.5.3. The two orbits of order-ideal rowmotion on $\mathcal{J}([2] \times [3])$.

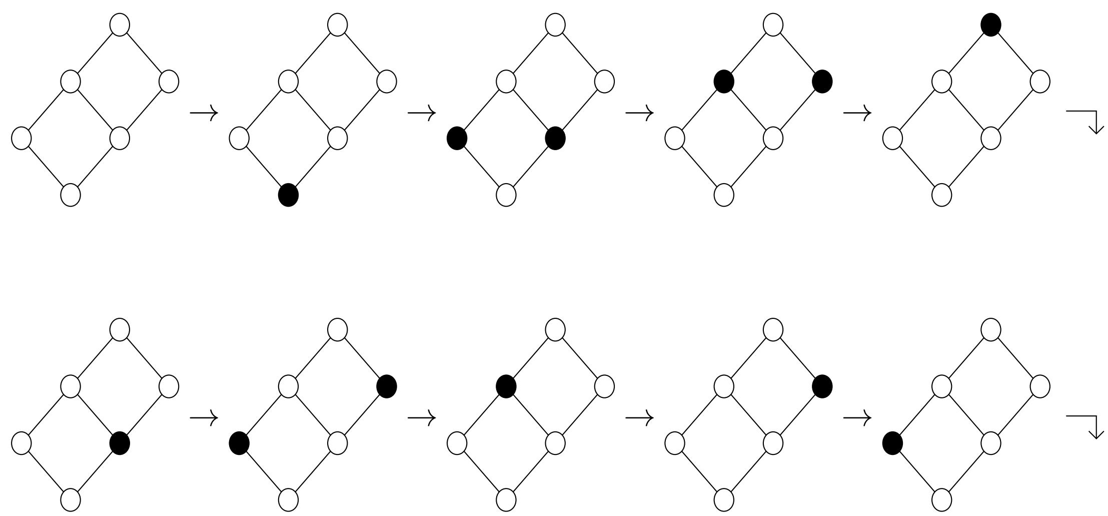


Adding up the number of filled-in nodes in each orbit we get 15; thus, each orbit has average cardinality of 3.

Here is a similar result for antichain rowmotion on antichains of $\mathcal{A}([a] \times [b])$ from the same paper.

Theorem 2.5.4 ([27], Section 3.3, Theorem 27)**.** *The cardinality statistic is c-mesic under the action of rowmotion ρ_A on $\mathcal{A}([a] \times [b])$, with $c = ab/(a+b)$.*

Example 2.5.5. Here are the two orbits of antichain rowmotion on $\mathcal{A}([2] \times [3])$.



Adding up the number of filled-in nodes in each orbit we get 6; thus, each orbit has average cardinality of 6/5.

We will revisit this combinatorial setting later with a new equivariant bijection that reproves Theorem 2.5.2.

Chapter 3

Whirling

3.1 Whirling function between finite sets

In this section we will let $\mathcal{F} \subseteq [k]^{[n]}$ be a family of functions $f : [n] \to [k]$. For the rest of this paper, we use $\{1, \ldots, k\} = [k]$ to represent the congruence classes of $\mathbb{Z}/k\mathbb{Z}$ as opposed to the usual $\{0, 1, \ldots, k-1\}$. For fixed values of k and n, we can represent such functions either in *two-line* notation or *one-line* notation, e.g., $f = (2\ 1\ 3\ 4\ 4)$ or $f = 21344$ each represent the function $f \in [4]^{[5]}$ with $f(1) = 2$, etc.

Definition 3.1.1 ([18])**.** For $f \in \mathcal{F}$ we define the *whirl* $w_i : \mathcal{F} \to \mathcal{F}$ at index i as follows: repeatedly add 1 (modulo k) to the value of $f(i)$ until we get a function in $\mathcal{F}$.

Example 3.1.2. Let $\mathcal{F} = \{f \in [4]^{[5]} : f(1) \neq 2\}$. f we apply w_2 to $f = 21344$, adding 1 in the second position gives 22344, but this is not in $\mathcal{F}$. Adding 1 again in this position gives the result: $w_2(f) = 23344$.

We will now highlight some specific results from the paper where whirling was first introduced. Let $\mathrm{Inj}_m(n,k)$ be the set of *m-injective functions*, that is, functions $f : [n] \to [k]$ such that $\#f^{-1}(t) \leq m$ for all $t \in [k]$. Similarly, let $\mathrm{Sur}_m(n,k)$ be the set of *m-surjective functions*, that is, $f : [n] \to [k]$ such that $\#f^{-1}(t) \geq m$ for all $t \in [k]$. Note that injective functions are 1-injections and surjective functions are 1-surjectives. We also define the statistic $\eta_j = \#f^{-1}(\{j\})$.

Theorem 3.1.3 ([18])**.** *Fix $\mathcal{F}$ to be either $\mathrm{Inj}_m(n,k)$ or $\mathrm{Sur}_1(n,k)$ for a given $n, k, m \in \mathbb{P}$. Then under the action of $\mathbf{w} = w_n \circ w_{n-1} \circ \cdots \circ w_1$ on $\mathcal{F}$, η_j is $\frac{n}{k}$-mesic for any $j \in [k]$*

The same result is conjectured to hold for $\mathrm{Sur}_m(n,k)$, but is still open. The key idea of the proof is to partition any orbit board into *chunks*, each of which contains each element of $[k]$ exactly once. Each chunk is constructed inductively starting from some 1 in the orbit board. If the element directly below is not a 2, then there must have been (and still be) a 2 elsewhere in the row or the row below. Add that 2 to the chunk. Continue this process starting at 2 then 3 and so on. Once every 1 is in a chunk, every other number is also in a chunk. Details on the proof can be found in Sections 2.2-2.4 of [18].

Example 3.1.4. Here is an orbit of $\mathbf{w}$ on $\mathrm{Inj}_1(3,6)$ containing $f = 415$.

$$415 \xrightarrow{\mathbf{w}} 621 \xrightarrow{\mathbf{w}} 342 \xrightarrow{\mathbf{w}} 563 \xrightarrow{\mathbf{w}} 124 \xrightarrow{\mathbf{w}} 356 \xrightarrow{\mathbf{w}} 412 \xrightarrow{\mathbf{w}} 534 \xrightarrow{\mathbf{w}} 651 \xrightarrow{\mathbf{w}} 263 \;\overset{\mathbf{w}}{\lrcorner}$$

And here is the orbit board partitioned into chunks:

4	1	5
6	2	1
3	4	2
5	6	3
1	2	4
3	5	6
4	1	2
5	3	4
6	5	1
2	6	3

Notice that each value $1, 2, \ldots, 6$ appear exactly 5 times in this orbit of size 10, in accordance with the 1/2-mesy of Theorem 3.1.3.

3.2 Order-reversing maps on posets

Let P be a poset with $n = \#P$, and let L be a bijection from P to $[n]$, called a *labeling* of P. A P-partition is a map σ from P to $\mathbb{N}$ such that if $x <_P y$, then $\sigma(x) \geq \sigma(y)$ [31, Definition 3.15.1].

Definition 3.2.1. An *order-reversing map* with maximum part k is a function $f : P \to [0, k]$ such that if $x \leq_P y$, then $f(x) \geq f(y)$. Let $\mathcal{F}_k(P)$ be the set of all such functions.

Recall that $\mathcal{L}(P)$ denotes the set of order-preserving maps from P to [n].

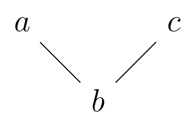
Definition 3.2.2. Let P be a poset. For $f \in \mathcal{F}_k(P)$ and $x \in P$, define $w_x : \mathcal{F}_k(P) \to \mathcal{F}_k(P)$, called the *whirl at x*, as follows: repeatedly add 1 (mod $k+1$) to the value of $f(x)$ until we get a function in $\mathcal{F}_k(P)$. This new function is $w_x(f)$.

Proposition 3.2.3. *If $x, y \in P$ are incomparable, then $w_x w_y(f) = w_y w_x(f)$.*

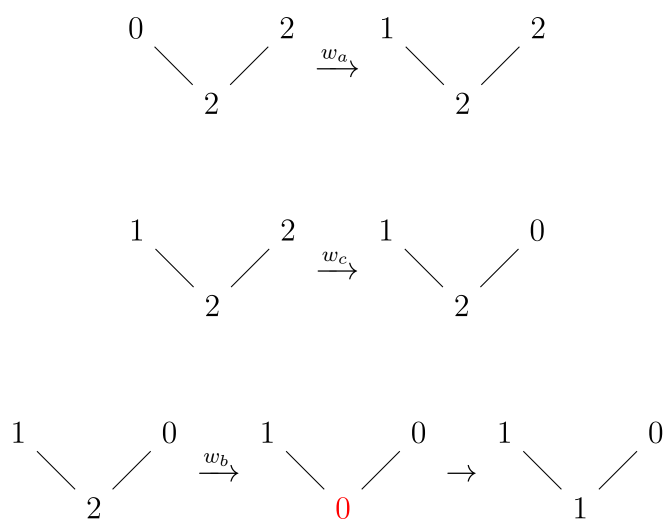
Proof. Since x and y are incomparable, there are no inequalities constraining the relationship between $f(x)$ and $f(y)$. So $w_x w_y = w_y w_x$. $\square$

Definition 3.2.4. For any linear extension ℓ of P, the elements of P are labeled $x_1, \ldots, x_n$. We define the *whirl* $w : \mathcal{F}_k(P) \to \mathcal{F}_k(P)$ to be the product $w = w_n w_{n-1} \cdots w_1$. We say $w(f)$ or just wf is the *whirl of* f.

Example 3.2.5. Let P be the three-element poset with nodes $\{a, b, c\}$ and the two relations $b \leq a$ and $b \leq c$.



Start with the labeling $f \in \mathcal{F}_2(P)$ such that $f(a) = 0$, $f(b) = 2$ and $f(c) = 2$. We use the linear extension (b, c, a) of P to compute $wf = w_b w_c w_a f$. First, add 1 to $f(a) = 0$ to get 1 at a, since $b \leq_P a$ and $f(a) \leq f(b)$ we stop with $w_a f(a) = 1$. Similarly, add 1 to $w_a f(c) = 2$ to get 0 so $w_c w_a f(c) = 0$. Finally add 1 to $w_c w_a f(b) = 2$ to get 0, but $w_c w_a f(a) = 1$ so add 1 one more time to get 1. We end with $wf(a) = 1$, $wf(b) = 1$ and $wf(c) = 0$.



There is a bijection between order ideals of a poset P and order-reversing maps in $\mathcal{F}_1(P)$. Specifically, an order-reversing map in $\mathcal{F}_1(P)$ is simply the indicator function of an order ideal $I \in J(P)$. We extend this to an equivariant bijection $\mathcal{F}_k(P) \to \mathcal{J}(\mathcal{P} \times [k])$ which sends w to ρ_J, meaning the following diagram commutes.

$$\begin{tikzcd} \mathcal{F}_k(P) \arrow[r, "w"] \arrow[d, leftrightarrow] & \mathcal{F}_k(P) \arrow[d, leftrightarrow] \\ \mathcal{J}(P \times [k]) \arrow[r, "\rho_J"] & \mathcal{J}(P \times [k]) \end{tikzcd}$$

We will call the chains $\{(x,1),(x,2),\ldots,(x,k)\} \subseteq P \times [k]$, for $x \in P$, the *fibers* of $P \times [k]$, and construct an equivariant bijection that first sends w_x to order-ideal toggling down the fiber $\{(x,1),(x,2),\ldots,(x,k)\}$.

Lemma 3.2.6. *There is an equivariant bijection between $\mathcal{F}_k(P)$ and $\mathcal{J}(P \times [k])$ which sends w_x to the toggle product $\tau_{(x,1)}\tau_{(x,2)}\ldots\tau_{(x,k)}$.*

Proof. First, we will establish a bijection ϕ between $\mathcal{F}_k(P)$ and $\mathcal{J}(P \times [k])$. Given $f \in \mathcal{F}_k(P)$, we will construct an order ideal $J(f) \in \mathcal{J}(P \times [k])$ by the following rule: $(x,i) \in J(f)$ if and only if $f(x) \leq i$. We claim the map $\phi(f) = J(f)$ is a bijection. The injective property of ϕ comes from the fact that $J(f) = J(g)$ would imply, for each fiber $\{(x,1),(x,2),\ldots,(x,k)\}$ there is maximum i_x such that $\{(x,1),(x,2),\ldots,(x,i_x)\} \subset J(f) = J(g)$, which gives us $f(x) = g(x) = i_x$ for each $x \in P$. To see that ϕ is surjective, let J be an order ideal of $P \times [k]$. Again, for each fiber $\{(x,1),(x,2),\ldots,(x,k)\}$ there is maximum i_x such that $\{(x,1),(x,2),\ldots,(x,i_x)\} \subset J$. Define $f_J : P \to [0,k]$ to be the map such that $f_J(x) = i_x$. We will show is $f_J \in \mathcal{F}_k$. Suppose not, i.e., that $f_J(x) > f_J(y)$ where $x >_P y$. This would imply that $(x, f_J(x)) \in J$ and $(y, f_J(x)) \notin J$. But J is an order ideal and $(x, f_J(x)) >_{P\times[k]}$

$(y, f_J(x))$, thus a contradiction, and it must be that $f_J \in \mathcal{F}_k$.

We will now complete the proof by fixing f and x and showing

$$\phi(w_x(f)) = \tau_{(x,1)}\tau_{(x,2)} \dots \tau_{(x,k)}(\phi(f)).$$

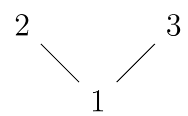
Let $m = \max(\{f(y) : y \lessdot x,\ \text{for } y \in P\} \cup \{0\})$ and $M = \min(\{f(y) : x \lessdot y,\ \text{for } y \in P\} \cup \{k\})$ so $m \leq f(x) \leq M$. There are two cases:

- Case 1. If $f(x) < M$, then $(w_x(f))(x) = f(x) + 1$.
- Case 2. If $f(x) = M$, then $f(x) + 1$ is greater than M so we increment through values modulo $k + 1$ until we get to $(w_x(f))(x) = m$.

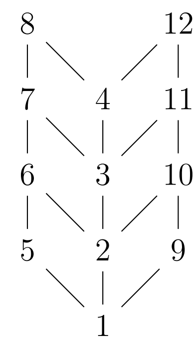
On the other hand, we consider the action of $\tau = \tau_{(x,1)}\tau_{(x,2)} \dots \tau_{(x,k)}$ on $\phi(f)$. Recall that $\tau_{(x,i)}$ does not change the order ideal whenever (x, i) is not maximal in the order ideal or minimal in the complement of the order ideal. Because of this there are only two outcomes of $\tau(S(f))$, either one element was added to top of the order ideal, or elements were removed until removing anymore would result in a non order ideal. But this lines up with the two cases above, which completes the proof. $\square$

Theorem 3.2.7. *Let P be a finite poset with some linear extension $\ell \in \mathcal{L}(P)$. There is an equivariant bijection between order-reversing functions $\mathcal{F}_k(P)$, and order ideals of $P \times [k]$, $\mathcal{J}(P \times [k])$, which sends whirling, $w = w_{x_1}w_{x_2} \cdots w_{x_n}$, to rowmotion on $\mathcal{J}(P \times [k])$, ρ_J.*

Proof. Since ρ_J can be expressed as a product of toggles which go down a linear extension, we will construct a linear extension on $P \times [k]$ and use Lemma 3.2.6. We define a linear extension ℓ_k on $P \times [k]$ as $\ell_k(x, i) = k(\ell(x) - 1) + i$. Here is an example linear extension ℓ on a poset P element poset and the corresponding linear extension ℓ_4 on $P \times [4]$.



ℓ



ℓ_4

Using ℓ_k rowmotion can be expressed as a product of toggles which go down the saturated chains of $P \times [k]$. By Lemma 3.2.6 there is an equivariant bijection from a products of toggles on these saturated chains to whirling at an element of P. The product of these whirls goes down the linear extension ℓ and thus is the whirl on order-reversing functions of P as desired. $\square$

As stated before, a consequence of this theorem is that the definition of whirl is well-defined for any linear extension of P. This is because any linear extension of P will give us a linear extension of $P \times [k]$ as seen in the proof of the theorem, and the order in which we toggle order-ideal rowmotion does not matter.

Similar to jump snakes from Lemma 1.2.10 or chunks from Theorem 3.1.3 we can partition orbits of w in neatly snakes.

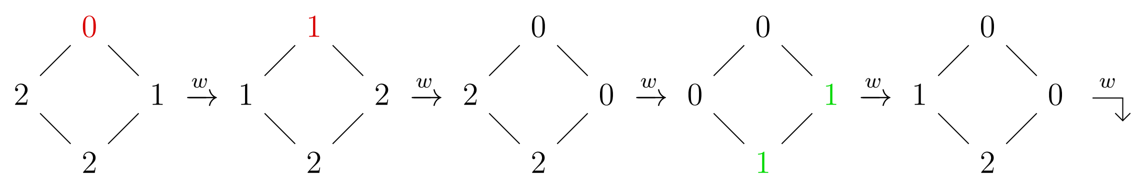
Definition 3.2.8. Let P be finite poset. For any $x \in P$ and $f \in \mathcal{F}_k(P)$, define (x, f) to be a *whirl element*. The whirl element (y, g) is *whirl successive* of (x, f) if either:

1. $y = x$ and $g(y) = w(f(x)) = f(x) + 1$, or

2. x covers y, $f = g$, and $f(x) = g(y)$.

We can think of whirl successive elements as being whirl elements who are one step away from each other. Either moving down the poset one element or whirling the function takes you from the whirl element (x, f) to the successive whirl element (y, g).

Example 3.2.9. Here is a w-orbit of $\mathcal{F}_2([2] \times [2])$



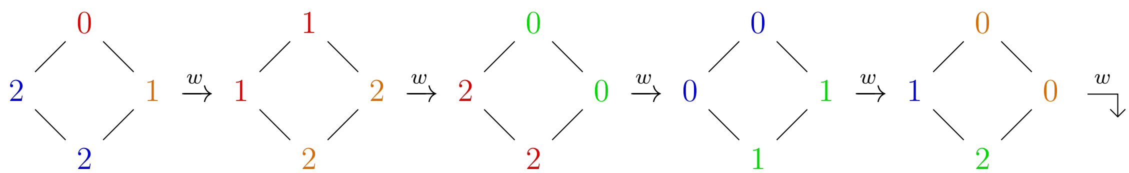
The two red-highlighted labels are whirl successive because they are in the same position in the poset but the second red label is one larger. The two green-highlighted labels are also whirl successive because one element covers the other and they both have the same label.

Two whirl elements (x, f) and (y, g) are *snake connected* if there exists a sequence of whirl successive elements $\{(x, f) = (x_0, f_0), (x_1, f_1), \ldots, (x_p, f_p) = (y, g)\}$.

Definition 3.2.10. A *snake* is a maximal set of snake connected whirl elements, that is, if (x, f) is in a snake and (x, f) is snake connected to (y, g), then (y, g) is in the snake.

Example 3.2.11. Using the orbit from the previous example we expand the snake successive elements to highlight the four snakes of the whirling orbit board. In this case, one can think of each snake starting at a 0 in the top element and ending at a

2 in the bottom element.

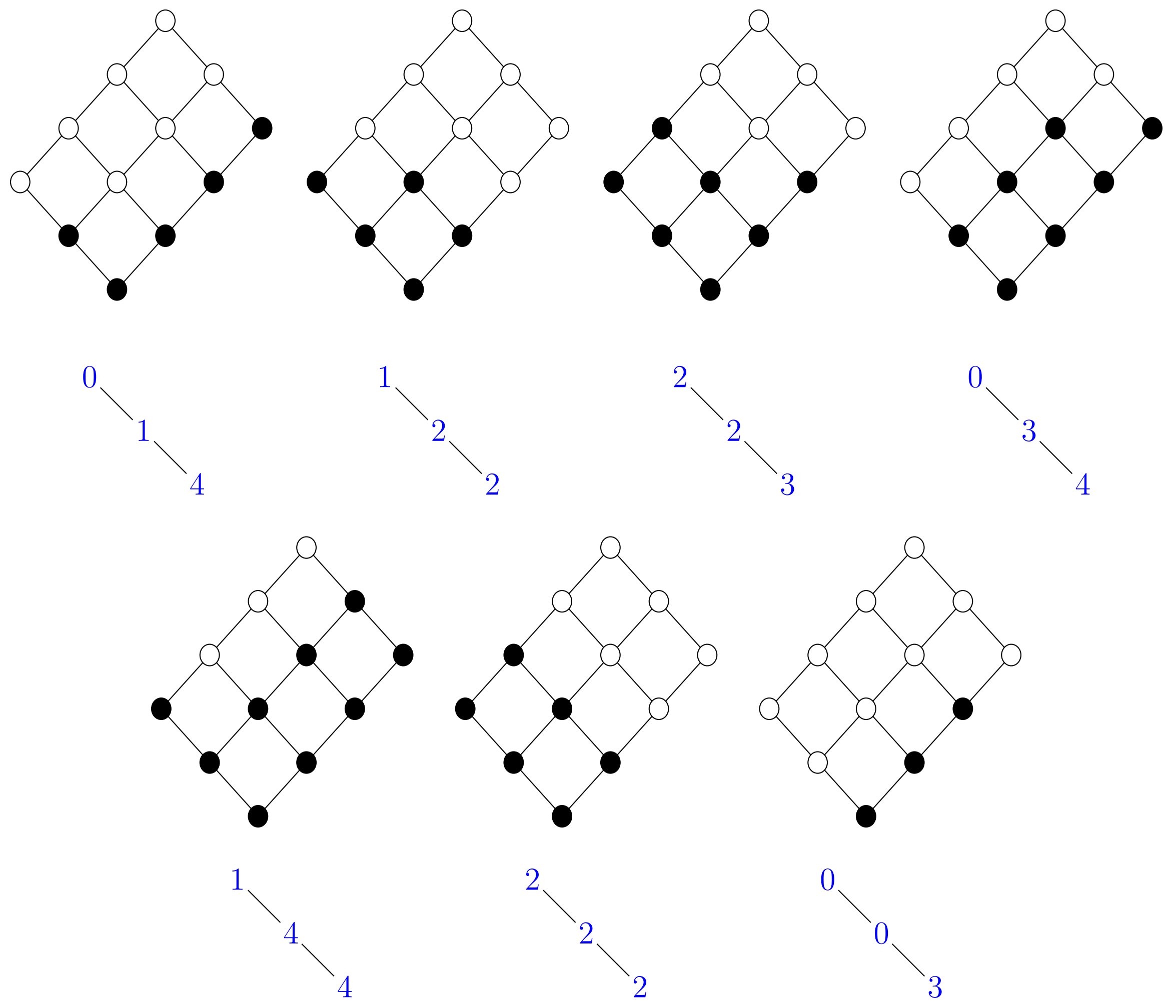


Partitioning orbits into snakes will help us prove Theorem 4.1.5, as well as give a new proof of Theorem 2.5.2.

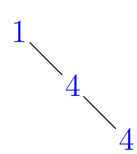
3.3 Revisiting homomesy for ρ_J on $[a] \times [b]$

Theorem 3.2.7 gives an equivariant bijection between $\mathcal{J}([a] \times [b])$ and $\mathcal{F}_b([a])$ carrying ρ_J to w. In this section we will give another proof for Theorem 2.5.2 by partitioning orbit boards into snakes given by Definition 3.2.10. We will also prove a new result on the snake decomposition itself. We begin with an example that demonstrates the bijection as created in the proof of Theorem 3.2.7.

Example 3.3.1. Here is an orbit of order-ideal rowmotion on $\mathcal{J}([3] \times [4])$ and below each order ideal is the associated order-reversing function. The order-reversing function is essentially counting the number of order-ideal elements in each fiber of $[3] \times [4]$.



The order reversing maps of $\mathcal{F}_b[a]$ will be written for convenience in one-line notation as $f(a)f(a-1)\dots f(1)$ where $f(a) \leq f(a-1) \leq \cdots \leq f(1)$. For example



$\rightarrow 144$

Example 3.3.2. Here is an orbit of $\mathcal{F}_4[3]$. Each row is an order-reversing function where the row after is the whirl from left to right of the row before it. One should think of an orbit board as being on a cylinder so the whirl of the last row is the first row.

$$\begin{array}{cc} & \xrightarrow{\;\leq\;} \\ w\Bigg\downarrow & \begin{array}{ccc} 0 & 0 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \\ 0 & 3 & 4 \\ 1 & 4 & 4 \\ 2 & 2 & 2 \end{array} \end{array}$$

Given an orbit board $\mathcal{O}$ of $\mathcal{F}_b[a]$ under the action of w, we can partition $\mathcal{O}$ into snakes according to Definition 3.2.10.

Lemma 3.3.3. *For an orbit board of $\mathcal{F}_k[n]$ with whirl element, (x, f), exactly one of the following holds:*

1. $wf(x) = f(x) + 1$, *or*

2. $f(x - 1) = f(x)$.

Proof. Suppose we have a whirl element (x, f) in an orbit board of $\mathcal{F}_k[n]$. If $wf(x) = f(x) + 1$, then we are done. If not, then $wf(x) + 1 > wf(x - 1)$. But since $f(x) \leq f(x - 1)$ it must be that $wf(x) = wf(x - 1)$. $\square$

We can interpret this lemma as giving us the two ways a snake can move through an orbit board of $\mathcal{F}_k[n]$. At every step it either moves down (increasing the value by one), or to the right (to a lower poset element). Looking at the orbit board above you will notice at any number i either i is to the right of it or $i + 1$ is below it.

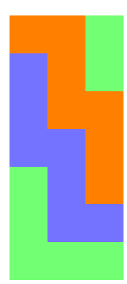
Lemma 3.3.4. *For any orbit board of the whirling action on $\mathcal{F}_b([a])$, the length of each snake is $a + b$.*

Proof. Since the snake contains (a, f) where $f(a) = 0$, we will use this as the snakes starting square. The snake will either have a square to the right or below in the orbit board. This process continues until it ends at $(1, w^b(f))$, where $w^b(f)(1) = b$. Thus the snake will effectively move $a-1$ places to the right in the orbit board and b places down for a total of $a+b$ elements in the snake. $\square$

Example 3.3.5. Below is the orbit board of $003 \in \mathcal{F}_4([3])$. The board is partitioned into 3 snakes of length 7.

0	0	3
0	1	4
1	2	2
2	2	3
0	3	4
1	4	4
2	2	2

We can think of snakes in this setting as either moving right or down the orbit board. When a snake moves down, the label inside the boxes increases, but when we move to the right, the label stays the same. Because of this, we don't need to write the numbers in the boxes once the snakes are outlined. The number in the box will always be the same as the number of rows the snake has moved down the orbit board. This picture alone can recover an entire orbit of rowmotion on order ideals of $[3] \times [4]$.



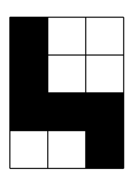
3.3.1 Right-down snakes

We now take a brief detour to examine these numberless snakes that show up in the orbit from the last example. In this subsection we will see a connection to generalized leftward cyclic shift on binary strings as defined in Definition 3.3.8. This section could be written as a study of lattice walks with north steps and east steps, but to make the connection clear we will call these lattice walks snakes even outside of the orbit board setting. A natural avatar of lattice walks with two directions is binary strings.

Definition 3.3.6. We define a *right-down snake* to be a walk on a chess board of length m and width k from the top left, $(1, 1)$, of the board to the bottom right, (k, m), using only unit length down and right steps. We denote each walk with a word composed of $k - 1$ $\mathbf{r}$'s and $m - 1$ $\mathbf{d}$'s. Each $\mathbf{r}$ represents a right step and each $\mathbf{d}$ represents a down step. Let $\mathcal{S}_{k\times m}$ be the set of all such words.

By Lemma 3.3.4 this definition lines up with the snakes in the whirling orbit board of $\mathcal{F}_a([b])$. To make that connection more concrete, we showed from the starting position of a snake there were b steps down and $a - 1$ steps to the right. So the snakes from $\mathcal{F}_a([b])$ are in bijection with the snakes from $\mathcal{S}_{a\times b+1}$. For the rest of this subsection we use the term "snake" to mean right-down snake. The previous definition gives us two ways to represent a snake, either as a block diagram on a chess board or as an element of $\mathcal{S}_{k\times m}$. For any $\varsigma \in \mathcal{S}_{k\times m}$ let $\varsigma(i)$ be the ith element in ς. For example if $\varsigma = \mathbf{rdrrdr}$, then $\varsigma(2) = \mathbf{d}$.

Example 3.3.7. The snake $\varsigma = \mathbf{ddrrd}$ has the following block diagram.



It is not hard to see that $\#\mathcal{S}_{k\times m} = \binom{m+k-2}{k-1}$, since of the $m+k-2$ elements in the word definition of a snake, we choose $k-1$ of them to be $\mathbf{r}$.

Recently, a simple generalization of leftward cyclic shift on binary strings was introduced by Hanaoka and Sadahiro [17].

Definition 3.3.8. [17] Fix $j \leq n$. Let $\{0,1\}^n$ be the set of binary strings of length n and fix $j \leq n$. We define $C_j : \{0,1\}^n \to \{0,1\}^n$ as follows: Let $s = s_0 s_1 \ldots s_{n-1} \in \{0,1\}^n$ be a binary string of length n. Set

$$C_j(s) = \begin{cases} s_{p+1}s_{p+2}\ldots s_{n-1}s_p \ldots s_1 & \text{if } p < j, s_w = 0 \text{ and } s_i = 1 \text{ for all } i < p \\ s_j s_{j+1} \ldots s_{n-1} s_{j-1} \ldots s_1 & \text{otherwise.} \end{cases}$$

Example 3.3.9. Consider $1110000010 \in \{0,1\}^{10}$. The first three entries are 1 with the fourth entry being 0, thus, for all $j \geq 4$, they are reversed and put at the end of the string,

$$C_j(1110000010) = 0000100111.$$

On the otherhand, if $j = 2$, then we just move the first 2 1's over,

$$C_2(1110000010) = 1000001011.$$

Notice that when $j = 1$ we recover the usual definition of bitstring rotation. This generalization comes from a generalization of independent sets called *j-independent sets* of the path graph. A set is j-independent if there are at least j 0's separating each pair of 1's. We consider C_m to act on elements of $\mathcal{S}_{m\times k}$, where $r \leftrightarrow 0$ and $d \leftrightarrow 1$.

Example 3.3.10. Fix $\varsigma = \mathbf{ddrrd} \in \mathcal{S}_{3\times 4}$. The first $\mathbf{r}$ shows up in the 3rd position so we reverse the first three letter and place them on the end of the word. Therefore we see

$$C_4(\mathbf{ddrrd}) = \mathbf{rdrdd}.$$

Under this action we get the symmetric homomesy on snakes similar to Theorem 1.2.8, but it requires less work to show under the action of generalized leftward cyclic shift. Let $\chi_i(\varsigma) = 1$ if the ith element in the word is $\mathbf{r}$ for all $i \in [m+k-2]$.

Theorem 3.3.11. *Fix positive integers k and m and let C_m act on $\mathcal{S}_{k\times m}$ via Definition 3.3.8. Then the statistic $\chi_i - \chi_{m+k-1-i}$ is 0-mesic with respect to the action of C_m.*

A stronger version of Theorem 3.3.11 is given in [17, Theorem 2]. We will prove this weaker version.

Proof. It suffices to show there is a bijection between the r's that show up in the ith position and the r's that show up in the $m+k-1-i$th position. Fix a snake $\varsigma \in \mathcal{S}_{k\times m}$ and suppose $\varsigma(i) = \mathbf{r}$. Let j be the number of $\mathbf{r}$'s to the left of the ith position. Let $\varsigma' = C_m^{j+1}(\varsigma)$ and we will argue $\varsigma(m+k-1-i) = \mathbf{r}$. In the case where $j = 0$, $\varsigma(i)$ is the first $\mathbf{r}$ on the left so $\varsigma' = C_m(\varsigma)$ moves that $\mathbf{r}$ i positions to the right but this means $\varsigma(m+k-1-i) = \mathbf{r}$. Now assume this holds for up to j $\mathbf{r}$'s before the ith position. Since this $\mathbf{r}$ is the $j+1$st $\mathbf{r}$, let w be the position of the jth $\mathbf{r}$ in ς. By assumption $C_m^{j+1}\varsigma(m+k-1-w) = \mathbf{r}$, then applying C_m one more time will push this $\mathbf{r}$ $i-w$ positions to the left, but this gives us $C_m^{j+2}\varsigma(m+k-1-i) = \mathbf{r}$ as desired. □

Example 3.3.12. Here is an example of this homomesy in the case where $k = 3$ and $m = 5$. The bottom row is the number of $\mathbf{r}$'s in each column. Notice that the bottom rows are a palindrome.

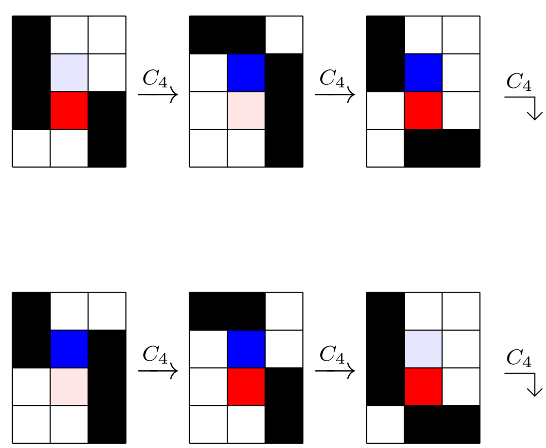
$$\begin{array}{cccccc} \mathbf{r} & \mathbf{r} & \mathbf{d} & \mathbf{d} & \mathbf{d} & \mathbf{d} \\ \mathbf{r} & \mathbf{d} & \mathbf{d} & \mathbf{d} & \mathbf{d} & \mathbf{r} \\ \mathbf{d} & \mathbf{d} & \mathbf{d} & \mathbf{d} & \mathbf{r} & \mathbf{r} \\ \hline 2 & 1 & 0 & 0 & 1 & 2 \end{array} \qquad \begin{array}{cccccc} \mathbf{r} & \mathbf{d} & \mathbf{r} & \mathbf{d} & \mathbf{d} & \mathbf{d} \\ \mathbf{d} & \mathbf{r} & \mathbf{d} & \mathbf{d} & \mathbf{d} & \mathbf{r} \\ \mathbf{d} & \mathbf{d} & \mathbf{d} & \mathbf{r} & \mathbf{r} & \mathbf{d} \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \end{array} \qquad \begin{array}{cccccc} \mathbf{r} & \mathbf{d} & \mathbf{d} & \mathbf{r} & \mathbf{d} & \mathbf{d} \\ \mathbf{d} & \mathbf{d} & \mathbf{r} & \mathbf{d} & \mathbf{d} & \mathbf{r} \\ \mathbf{d} & \mathbf{d} & \mathbf{r} & \mathbf{r} & \mathbf{d} & \mathbf{d} \\ \hline 1 & 0 & 2 & 2 & 0 & 1 \end{array}$$

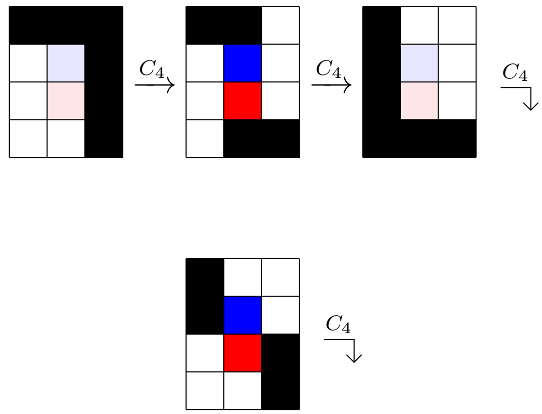
$$\begin{array}{cccccc} \mathbf{d} & \mathbf{r} & \mathbf{d} & \mathbf{r} & \mathbf{d} & \mathbf{d} \\ \mathbf{d} & \mathbf{r} & \mathbf{d} & \mathbf{d} & \mathbf{r} & \mathbf{d} \\ \mathbf{d} & \mathbf{d} & \mathbf{r} & \mathbf{d} & \mathbf{r} & \mathbf{d} \\ \hline 0 & 2 & 1 & 1 & 2 & 0 \end{array} \qquad \begin{array}{cccccc} \mathbf{d} & \mathbf{r} & \mathbf{r} & \mathbf{d} & \mathbf{d} & \mathbf{d} \\ \mathbf{r} & \mathbf{d} & \mathbf{d} & \mathbf{d} & \mathbf{r} & \mathbf{d} \\ \mathbf{d} & \mathbf{d} & \mathbf{d} & \mathbf{r} & \mathbf{d} & \mathbf{r} \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

We conjecture a stronger version of this homomesy for the block diagrams.

Conjecture 3.3.13. *Two positions on a board are* symmetrically placed *if one is in the place of the other after 180-degree rotation. Symmetric placed positions on a board are visited by a right-down snake the same number of times during an orbit of* C_m.

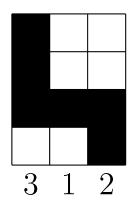
Example 3.3.14. For each orbit of C_4, the red square is part of a snake the same number of times the blue square is.





There is a natural bijection from snakes to compositions of $m+k-1$ of length k where each component represents the number of snake elements in a column of the block diagram. As it stands a direct relation is not known, but this looks similar to work in [2, Theorem 6.5].

Example 3.3.15. Given the snake $\varsigma = \mathbf{ddrrd} \in \mathcal{S}_{3\times 4}$ we get the composition $(3, 1, 2)$.



This composition can also be obtained by counting the number of **d**'s between successive **r**'s and adding one.

$$\color{red}{\mathbf{dd}}\color{black}{\mathbf{rr}}\color{blue}{\mathbf{d}}\color{black} \to (\color{red}{3}\color{black}, 1, \color{blue}{2}\color{black})$$

Since the map C_m reverses the first set of elements of $\varsigma \in \mathcal{S}_{k\times m}$ up to the first **r** and places them at the end of the word, it makes senses that the effect of C_m through the bijection to compositions is leftward cyclic shift.

Lemma 3.3.16. *Fix k, m for $\mathcal{S}_{k\times m}$. Let c be leftward-cyclic shift on compositions. There is an equivariant bijection between $\mathcal{S}_{m\times k}$ and compositions of $m+k-1$ of length k which sends C_m to c.*

Proof. We have established a bijection between $\mathcal{S}_{m\times k}$ and compositions of $m+k-1$ of length k defined explicitly in Example 3.3.15. There are only $m-1$ **d**'s in each element of $\mathcal{S}_{m\times k}$ so, we will always be in the initial case of the piecewise definition of C_m given in Definition 3.3.8. Thus, C_m will always have the effect of taking the first string of **d**'s and first r, flipping them and placing them at the end of the string. This results in leftward cyclic shift on the composition level as desired. $\square$

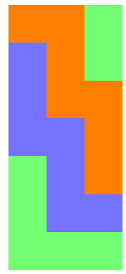
Example 3.3.17. Let $\varsigma = \mathbf{ddrrd} \in \mathcal{S}_{3\times 4}$.

$$\begin{tikzcd} \mathbf{ddrrd} \arrow[r, "C_4"] \arrow[d, leftrightarrow] & \mathbf{rdrdd} \arrow[d, leftrightarrow] \\ (3,1,2) \arrow[r, "c"] & (1,2,3) \end{tikzcd}$$

Now we remind ourselves that these snakes come from orbit board of $\mathcal{F}_b([a])$, with our new interpretations we can notice something nice about the orbit boards that would have been missed otherwise.

Example 3.3.18. An orbit board of $\mathcal{F}_4([3])$. The orbit contains three snakes, $(1,3,3)$

in orange, $(3, 3, 1)$ in blue, and $(3, 1, 3)$ in green.



This motives the next theorem, which states that the snakes of a whirling orbit board of $\mathcal{F}_b([a])$ are cyclic shifts of each other.

Definition 3.3.19. Let ς_1 and ς_2 be snakes in an orbit board of $\mathcal{F}_b([a])$. We say ς_2 is *in front of* ς_1 if starting at a position in ς_1 not in the first column and moving diagonally down-left one spot in the orbit board is a position in ς_2.

So using the last example, the blue snake is in front of the orange snake, which is in front of the green snake, which is in front of the blue snake.

Lemma 3.3.20. *Let ς_1 and ς_2 be snakes in an orbit board of $\mathcal{F}_b([a])$ such at ς_2 is in front of ς_1. Then $\varsigma_2 = C_a(\varsigma_1)$.*

Proof. Take a position in ς_1 and move down and to the left, that spot is in ς_2. This tells us the first $a - 1$ columns of ς_2 are the last $a - 1$ columns of ς_1 shifted down. Finally fill in the last column with what is need to be length $a + b$, but this is exactly what was in the first column. This description is leftward cyclic shift. □

Now that we have connected the snakes of an orbit board of $\mathcal{F}_b([a])$ and leftward cyclic shift on compositions, we have a limit on how large any orbit can be.

Corollary 3.3.21. *The number of snakes in an orbit board of $\mathcal{F}_b([a])$ divides a.*

Proof. By Theorem 3.3.20 the number of snakes cannot be larger than the order of leftward cyclic shift on a composition. Since the width of the board is a, the number of snakes must divide a. $\square$

We will now focus our attention on reproving Theorem 2.5.2 by defining an action on snakes in an orbit board and showing this action does not change the cardinality of the orbit.

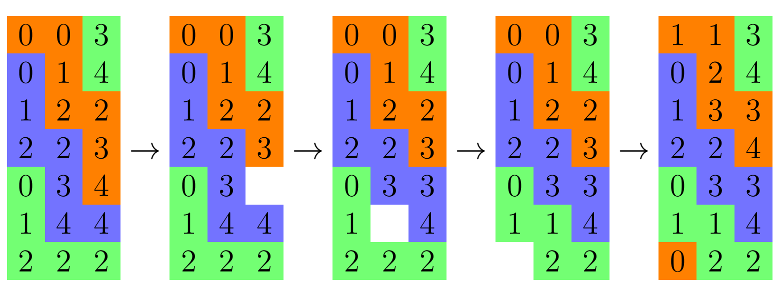
Definition 3.3.22. We define a process called *snake promotion* on an orbit board as follows:

1. Pick a snake with more than one element in the right-most column and call it ς. Delete the final element of ς. This creates a hole.

2. Move down-left diagonally from the hole and swap that position with the hole. Do this until the hole is in the leftmost column.

3. Since the hole has moved over $a-1$ snakes, the hole is now above ς. Add the hole back onto the ς at the beginning.

Example 3.3.23. Here is an example of snake promotion step by step. We will choose the orange snake to perform snake promotion on.



The orange snake started at $(1,3,3)$ and after snake promotion became $(2,3,2)$. Since this process removes a snake block from the last columns and adds one to the first, the effect of snake promotion on the compositions is subtracting 1 from the last position and adding 1 to the first position. Snake promotion also changes the other snakes in the orbit board as well. However, leftward cyclic shift on a composition of a snake give you another snake in the orbit board, so the effect of snake promotion on the other snakes is similar to that of the one snake promotion was preformed on.

Lemma 3.3.24. *Given an orbit of whirling on $\mathcal{F}_b([a])$, snake promotion does not change the sum of the labels of the orbit board.*

Proof. We will inspect what happens to the sum of the labels of an orbit board of $\mathcal{F}_b([a])$ step by step through snake promotion. Taking away the last element of ς reduces the sum of the all labels by b. Next swapping snake positions with the hole decreases the sum of all the labels by 1 each time for a net loss of $a-1$. Finally adding the block to the start of ς increases the labels all other snake elements by 1 for, resulting in a net gain of $b+a-1$. $\square$

To derive Theroem 2.5.2 from the lemma, it will suffice to show that for any orbit board there is a sequence of snake promotions taking us to an orbit board whose composition is almost all 1's.

Proof of Theorem 2.5.2. Start with an orbit board $\mathcal{O}$ of $\mathcal{J}([a]\times[b])$. Let ς be a snake in $\mathcal{O}$. We will follow this process:

1. Perform snake promotion until the rightmost entry of the composition is 1.

2. Perform leftward cyclic shift on the composition to get another snake in the orbit board.

3. Repeat steps 1 and 2 until the you have a snake with composition $(b+1, 1, \ldots, 1)$.

We complete this proof by showing an orbit with snake $(b+1, 1, \ldots, 1)$ has cardinality $(a+b)ab/2$. A generic snake with composition $(1, 1, \ldots, b+1, 1, \ldots, 1)$ where b is in the dth entry, in this orbit will have cardinality $1+2+\cdots+b$ from the steps downs and $a-d$ copies of b. There are a snakes so we have that the cardinality is

$$a\binom{b+1}{2} + b\binom{a}{2}.$$

But this simplifies to $(a+b)ab/2$. Since the total number of rows in one orbit is $(a+b)$, dividing by $(a+b)$ gives us the average $ab/2$. $\square$

Example 3.3.25. Start with the orbit in $\mathcal{F}_6([4])$ that contains the snake with composition $(3, 4, 1, 2)$. Perform snake promotion once to get $(4, 4, 1, 1)$. Perform leftward cyclic shift once on the composition to get $(4, 1, 1, 4)$. Perform snake promotion three times to get $(7, 1, 1, 1)$.

0	2	3	3
1	3	3	4
2	2	4	5
0	3	5	6
1	4	6	6
2	5	5	5
3	3	3	6
0	0	4	4
0	1	1	5
1	1	2	6

Blue snake: 33
Red snake: 31
Green snake: 25
Orange snake: 31
Total: 120

0	0	0	0
0	0	0	1
0	0	1	2
0	1	2	3
1	2	3	4
2	3	4	5
3	4	5	6
4	5	6	6
5	6	6	6
6	6	6	6

Blue snake: 21
Red snake: 27
Green snake: 33
Orange snake: 39
Total: 120

On the left is the orbit that has snake composition $(3, 4, 1, 2)$, and on the right is the orbit that has snake composition $(7,1,1,1)$. The sum of all the labels of each orbit board is $\frac{(6+4)(4)(6)}{2} = 120$.

It was observed by Brouwer and Schrijver [5] that order of rowmotion on $J([a]\times[b])$ is $a+b$. This can also be seen from the last Corollary by observing that the strings of length a being rotated are in fact compositions of $a+b$. Since these components are in bijection with the length of the snake segments, we get that the corresponding super orbit board must be of length $a+b$.

Corollary 3.3.26. *In an orbit of whirling functions of $\mathcal{F}_b([a])$, the average sum of the labels on the snakes is $b(a+b)/2$.*

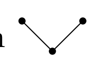
Proof. By Theorem 2.5.2 the sum of the labels over an entire super orbit board is $(a+b)ab/2$, and by Corollary 3.3.21 the number of snakes in a super orbit board is a. The result follows. $\square$

Chapter 4

Periodicity and homomesy for rowmotion on $\mathsf{V} \times [k]$

In this section we define the object of interest, a wedge poset cross a chain poset, and establish interesting homomesies for order-ideal rowmotion.

4.1 A wedge cross a chain poset

Definition 4.1.1. Let V be the wedge poset with Hasse diagram  and define $\mathsf{V}_k = \mathsf{V} \times [k]$, where $[k]$ is the chain poset.

Example 4.1.2. The Hasse diagram of V_4 is shown in Figure **??**.

We will often want to discuss specific nodes of these posets, so we establish the following labeling.

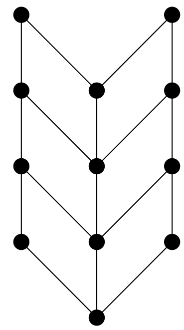
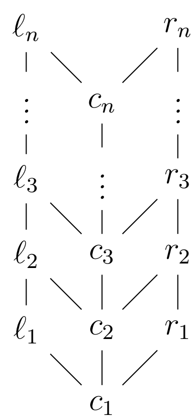


FIGURE 4.1: $\mathsf{V}_4 = \mathsf{V} \times [4]$

Notation 4.1.3. *We label the nodes in V_k as follows*



With just this notation, we are ready to state the main theorems of this section.

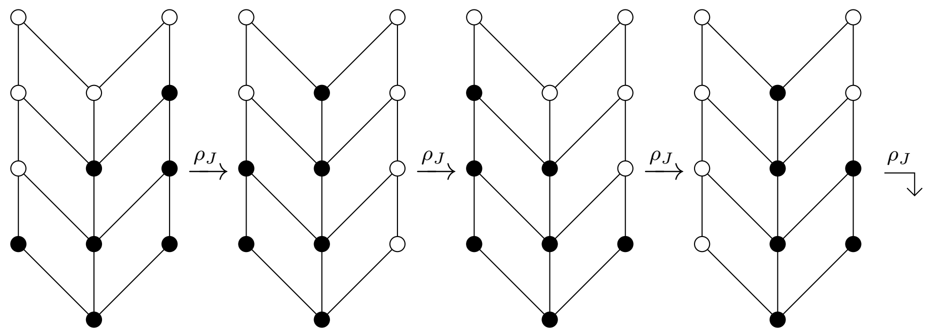
Theorem 4.1.4. *The order of ρ_J, order-ideal rowmotion, on $\mathcal{J}(\mathsf{V}_k)$ is $2(k+2)$.*

Theorem 4.1.5. *If χ_x is the indicator function for the node x, then for the action of ρ_J on $\mathcal{J}(\mathsf{V}_k)$*

1. *The statistic $\chi_{\ell_i} - \chi_{r_i}$ is 0-mesic for all $i \in \{0 \dots k\}$.*

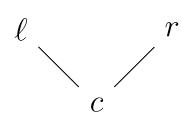
2. *The statistic $\chi_{\ell_1} + \chi_{r_1} - \chi_{c_k}$ is $\frac{2(k-1)}{k+2}$-mesic.*

Example 4.1.6. This is an example of an orbit of ρ_J on $\mathcal{J}(\mathsf{V}_4)$. Notice the size of the orbit, 4, divides $2(4+2) = 12$.



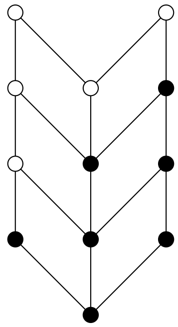
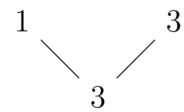
To prove these theorems we will construct an equivariant bijection from $\mathcal{J}(\mathsf{V}_k)$ to triples with a unique max center entry, instead of studying ρ_J on $\mathcal{J}(\mathsf{V}_k)$ directly.

Using Theorem 3.2.7 the order ideals of V_k are in bijection with $\mathcal{F}_k(\mathsf{V})$, the set of order-reversing functions. An order-reversing function $f = (\ell, c, r)$ on V assigns a value to each element of V as follows:



under the condition that $c \leq \ell$ and $c \leq r$. Furthermore, this bijection carries rowmotion on $\mathcal{J}(\mathsf{V}_\mathsf{k})$ to whirling down a linear extension of V, which is easily described in terms of certain triples.

We denote this bijection by ϕ. The bijection ϕ sends an order ideal I to a function (ℓ, c, r) by associating ℓ, c, and r with the number of elements of the order ideal in the left, center, and right fibers respectively.

 $\overset{\phi}{\longleftrightarrow}$ 

$\longleftrightarrow$ $(1,3,3)$

Example 4.1.7. Here is the orbit of $\mathcal{F}_4(\mathsf{V})$ containing $(1,3,3)$, matching Example 4.1.6:

$$(1,3,3) \xrightarrow{w} (2,4,0) \xrightarrow{w} (3,3,1) \xrightarrow{w} (0,4,2) \overset{w}{\lrcorner}$$

Proposition 4.1.8. *Let $|\mathcal{J}(\mathsf{V}_k)|$ be the number of order ideals of V_k.*

$$|\mathcal{J}(\mathsf{V}_k)| = \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Proof. We first count how many order-reversing functions f in $\mathcal{F}_k(\mathsf{V})$. For each value $f(c)$ there are $f(c)$ independent choices for $f(\ell)$ and $f(r)$ which result in a order-reversing function. By Theorem 3.2.7, $\mathcal{F}_k(\mathsf{V})$ is in bijection with $\mathcal{J}(\mathsf{V}_k)$. $\square$

4.2 Center-seeking snakes

In proving the order of ρ_J on $\mathcal{J}(\mathsf{V}_k)$ is $2(k+2)$ we end up proving something stronger, that is, that $\rho_J^{k+1}(I)$ results in a order ideal that has been reflected across the the center chain. To prove this we will investigate the snakes that arise from repeatedly whirling an order-reversing function.

Recall from Definition 3.2.10 that given a whirling orbit board, $\mathcal{O} = \{f, w(f), w^2(f), \ldots\}$ of w on $\mathcal{F}_k(V)$, a *snake* ς is a maximal set of snake connected elements. Here are two orbit boards of $\mathcal{F}_4(\mathsf{V})$, one with six snakes and one with two snakes. Notice that the snakes in the second orbit have two "starting" positions.

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

0	2	0
1	3	1
2	4	2
3	3	3
0	3	0
2	2	2

We can think about snakes as either starting on the left, or the right; or both left and right. We call the latter *two-tailed.* Since these snakes move down the orbit board at every step except for one, we have two useful interpretations for them. For one we think of these snakes as a sequence of function values in the orbit board which start at 0 and end at k, where one value is repeated when moving into the center. Since these snakes move from the outer columns of the orbit board to the center column, we can think of these snakes as *center-seeking snakes* to not confuse them with right-down snakes. Here we isolate one example of a left snake $\varsigma = (0, 1, 2, 2, 3, 4)$ visualized in a orbit board of $\mathcal{F}_4(\mathsf{V})$.

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

The first observation is that all snakes are of length $k+2$. This is clear since there are $k+1$ elements $0, ..., k$ and one of them will be doubled. We can also write a snake as a composition as we did in Section 3, but we would lose information about which side the snake started on. Define $cc(\varsigma)$ to be the number of blocks of the snake in the center column and $oc(\varsigma)$ to be the number of blocks of ς in outer columns. For snakes that begin on either the left or the right $oc(\varsigma) = (k+2) - cc(\varsigma)$.

Lemma 4.2.1. *Let S be the set of snakes in an orbit board $\mathcal{O}$ of w on $\mathcal{F}_k(V)$, and let $\#\mathcal{O}$ be the length of $\mathcal{O}$.*

$$\sum_{\varsigma \in S} oc(\varsigma) = 2\#\mathcal{O}$$

Proof. Since $oc(\varsigma)$ counts the number of blocks of ς not in the center column, summing over all snakes gives the number of blocks in the left and right column. $\square$

Similarly, if S_r is just the snakes that appear in the right most column, then

$$\sum_{\varsigma \in S_r} oc(\varsigma) = \#\mathcal{O}.$$

Example 4.2.2. Here is the previous example with all the snakes colored. Notice that the number of blocks in the right columns of the green, yellow, and teal snakes are $4, 5, 3$ respectively and the orbit board is of length 12.

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

In the setting of $\mathcal{F}_k(\mathsf{V})$, as long as we know the number of blocks of a snake in the outer column and which sides it started on, we can recover the entire snake. Now it would be helpful to repeat some of the process from right-down snakes, so we will define what it means for one snake to be in front of another snake.

Definition 4.2.3. We will place a circular order on the snakes. Let ς_1 and ς_2 be snakes in an orbit board of $\mathcal{F}_k(\mathsf{V})$. We say ς_2 is *in front of* ς_1 if starting at a position in ς_1 in the center column and moving down one position in the orbit board is a position in ς_2 or ς_1. When referring to a sequence of snakes that are in front of each other we will call them *consecutive*.

Example 4.2.4. Look back at Example 4.2.2. The red snake is in front of the green snake, and the yellow snake is in front of the red snake.

Lemma 4.2.5. *Assume an orbit board $\mathcal{O}$ of w on $\mathcal{F}_k(V)$ has no two-tailed snakes. Let ς_1, ς_2, and ς_3 be three consecutive snakes, that is, ς_3 is in front of ς_2 which is in* *front of* ς_1 *in* $\mathcal{O}$*. Then,*

$$oc(\varsigma_3) = 2(k+2) - (oc(\varsigma_1) + oc(\varsigma_2)).$$

Otherwise, if there are two-tailed snakes, then ς_2*,* $cc(\varsigma_1) + cc(\varsigma_2) = k+2$*.*

Proof. The third snake, ς_3, begins beneath ς_1, then ς_3 will go into the center when ς_2 terminates. So

$$oc(\varsigma_3) = cc(\varsigma_1)+cc(\varsigma_2) = (k+2-oc(\varsigma_1))+(k+2-oc(\varsigma_2)) = 2(k+2)-(oc(\varsigma_1)+oc(\varsigma_2)).$$

Now if our orbit contains two tailed snakes, then ς_2 needs only to wait for ς_1 to terminate so we get $cc(\varsigma_2) = k+2-cc(\varsigma_1)$. □

Example 4.2.6. In Example 4.2.2, the oc statistic is 5 for the blue snake, 4 for the green one, and 3 for the red one. We see $2(4+2)-(5+4)=3$.

Lemma 4.2.7. *Given an orbit board without two-tailed snakes, let* ς_1*,* ς_2*,* ς_3*, and* ς_4 *be consecutive, then*

$$oc(\varsigma_4) = oc(\varsigma_1)$$

Proof. If we apply Lemma 4.2.5 to $oc(\varsigma_4)$ we see

$$oc(\varsigma_4) = 2(k+2) - (oc(\varsigma_3) + oc(\varsigma_2)).$$

If we apply the lemma to ς_3, we get

$$oc(\varsigma_4) = 2(k+2) - (2(k+2) - ((oc(\varsigma_2) + oc(\varsigma_1))) + oc(\varsigma_2)),$$

which reduces to $oc(\varsigma_1)$. $\square$

Notice that we are not claiming the board starts to repeat, since by parity ς_4 will start on the opposite side of ς_1.

Lemma 4.2.8. *Let $\mathcal{O}$ be an orbit board of w on $\mathcal{F}_k(\mathsf{V})$. If $\mathcal{O}$ is without two-tailed snakes, then there are 6 unique snakes. If $\mathcal{O}$ has two-tailed snakes, then there are only two unique snakes.*

Proof. First suppose the orbit board does not contain two-tailed snakes. Let $\varsigma_1, \ldots, \varsigma_7$ be seven consecutive snakes in an orbit board of $\mathcal{F}_k(\mathsf{V})$. We will show $\varsigma_1 = \varsigma_7$. A consequence of Lemma 4.2.7 is $oc(\varsigma_i) = oc(\varsigma_{i+3})$. Since the parity of which side the snakes starts on changes for each consecutive snake, we have $oc(\varsigma_i) = oc(\varsigma_{i+6})$, and they start on the same side. So we know ς_i and ς_{i+6} are the same, giving us 6 unique snakes.

Now assume the orbit contains two-tailed snakes. Let $\varsigma_1, \ldots, \varsigma_3$ be three consecutive sankes in an orbit board of $\mathcal{F}_k(\mathsf{V})$. By Lemma 4.2.5, we know $cc(\varsigma_i) + cc(\varsigma_{i+1}) = k + 2$, so $cc(\varsigma_i) = cc(\varsigma_{i+2})$, and since two-tailed snakes start on both the left and the right, snakes ς_i and ς_{i+2} are the same, giving us 2 unique snakes. $\square$

Theorem 4.2.9. *The length of an orbit board of w on $\mathcal{F}_k(V)$ without two-tailed snakes is $2(k+2)$, and the length of an orbit board with two-tailed snakes is $k+2$.*

Proof. First assume $\mathcal{O}$ is without two-tailed snakes. Since there are 6 unique snakes, by Lemma 4.2.1 we have

$$oc(\varsigma_1) + oc(\varsigma_2) + oc(\varsigma_3) + oc(\varsigma_4) + oc(\varsigma_5) + oc(\varsigma_6) = 2\#\mathcal{O}.$$

Let $oc(\varsigma_i) = i$ and $oc(\varsigma_2) = j$, then by Lemma 4.2.5 we have

$$i + j + 2(k+2) - (i+j) + i + j + 2(k+2) - (i+j) = 2\#\mathcal{O}.$$

But this gives us

$$\#\mathcal{O} = 2(k+2).$$

The proof follows in a similar manner for orbits containing two-tailed snakes. □

Theorem 4.2.10. *Let* $(a, b, c) \in \mathcal{F}_k(\mathsf{V})$*, then* $w^{k+2}(a, b, c) = (c, b, a)$.

Proof. First assume $a = c$ so the orbit board has two-tailed snakes. By Lemma 4.2.8, we know there are exactly two snakes, call them ς_1 and ς_2. By Lemma 4.2.5, we know $cc(\varsigma_1) + cc(\varsigma_2) = k + 2$. Since there are only two snakes, $k + 2$ counts the number of labels in the center column of one orbit in the orbit board. Thus the orbit board of just one orbit will have exactly $k + 2$ rows. Since the orbit board has exactly $k + 2$ rows, $w^{k+2}(a, b, c) = (a, b, c)$, but $a = c$ so we have satisfied the conclusion of the theorem.

Now let $\mathcal{O}$ be the orbit board of w that contains (a, b, c) with $a \neq c$. Since $a \neq c$, the orbit board is without two-tailed snakes, so there will be a left snake that 'moves' into the center, that is, there is an element $(h, h, g) \in \mathcal{O}$ (as in the third row of the red snake in Example 4.2.2). Call the snake that passes through both h's, ς_1, and the snake that passes through g, ς_2. Let ς_3 and ς_4 be the next two consecutive snakes. By 4.2.5 we know if $oc(\varsigma_1) = i$ and $oc(\varsigma_2) = j$, then $oc(\varsigma_3) = 2(k + 2) - (i + j)$ and $oc(\varsigma_4) = i$. The number of iterations of w between (h, h, g) and the triple where ς_4 bends in is

$$k + 2 - oc(\varsigma_1) + k + 2 - oc(\varsigma_2) + k + 2 - oc(\varsigma_3).$$

Which simplifies to

$$3(k+2)-(i+j+2(k+2)-(i+j)=k+2.$$

It follows that the number of iterations between (h,h,g) and (g,h,h) is $k+2$. There is a power of w, say e, such that $w^e(a,b,c)=(h,h,g)$ thus,

$$w^{k+2}(a,b,c)=w^{-e}w^{k+2}w^e(a,b,c)=w^{-e}w^{k+2}(h,h,g)=w^{-e}(g,h,h)=(c,b,a).$$

$\square$

Now that we have covered periodicity, we will cover the previously mentioned homomesies.

Proof of Theorem 4.1.5. The statistic $\chi_{\ell_i}-\chi_{r_i}$ being 0-mesic comes quickly from Theorem 4.2.10. Since (a,b,c) and (c,b,a) both show up in the same orbit, there is a bijection between each ℓ_i in an orbit of order-ideal rowmotion and a corresponding r_i that shows up after $k+2$ iterations of rowmotion.

Now we will show $\chi_{\ell_1}(I)+\chi_{r_1}(I)-\chi_{c_k}(I)$ is $\frac{2(k-1)}{k+2}$-mesic. If I is an order ideal of $\mathsf{V}\times k$ in an orbit $\mathcal{O}$ of order-ideal rowmotion ρ_J, then there is a triple $(\ell,c,r)\in\mathcal{F}_k(\mathsf{V})$ in bijection with I given by Theorem 3.2.7. We know if $\ell_1\in I$, then $\ell>0$. Also if $r_1\in I$, then $r>0$. Finally if $c_k\in I$, then $c=k$. There are two types of orbits, those where $\ell=r$ and those where $\ell\neq r$. First assume $\ell\neq r$. We will investigate the orbit of whirling generated by (ℓ,c,r). By Lemma 4.2.8 we know there are there are 6 unique snakes in the orbit. By Theorem 4.2.9, we know the orbit of length $2(k+2)$, and thus so is each column of the orbit board, that gives us in total $4(k+2)$ entries in the left and right column with six of those being 0. So there are $4(k+2)-6$ entries that are not 0 in the left and right columns of the orbit board; thus, they contribute to either the number of r_1 or ℓ_1. The number of elements in the center column which are k is 6 because there are 6 snakes that end at k, all in the center. Therefore, $\sum_{I\in\mathcal{O}} \chi_{\ell_1} + \chi_{r_1} - \chi_{c_k} = 4(k+2) - 12$. We divide by the length of the orbit to get the average

$$\frac{\sum_{I\in\mathcal{O}} \chi_{\ell_1}(I) + \chi_{r_1}(I) - \chi_{c_k}(I)}{\#\mathcal{O}} = \frac{4(k+2)-12}{2(k+2)} = \frac{2(k-1)}{k+2}.$$

The case where $\ell = r$ follows similarly: By Lemma 4.2.8 and Theorem 4.2.9, there are 2 unique snakes in an orbit of whirling of length $k+2$. that give us a total $2k$ nonzero entries in the left and right columns. We subtract the two k's in the center column to get $\sum_{I\in\mathcal{O}} \chi_{\ell_1} + \chi_{r_1} - \chi_{c_k} = 2k - 2$. Divide by the length of the orbit to get the result. $\square$

4.3 Another homomesy

There is another homomesy which does not have an intuitive understanding from order ideals but is easily understood from snakes. Let I be an order ideal in $\mathcal{J}(\mathsf{V}\times\mathsf{k})$, and we define the statistic $f_i = \chi_{\ell_i} + \chi_{r_i} + \chi_{c_{i-1}}$, which has the *flux-capacitor* shape seen in Figure 4.2.

Lemma 4.3.1. *Let $f_i = \chi_{\ell_i} + \chi_{r_i} + \chi_{c_{i-1}}$. The difference $f_{i+1} - f_i$ is $\frac{3}{k+2}$-mesic under the action of rowmotion on order ideals of $\mathcal{J}(\mathsf{V}\times[k])$.*

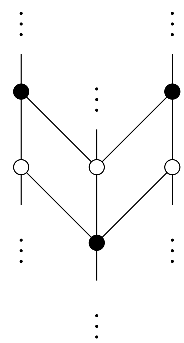


FIGURE 4.2: $f_i = \chi_{\ell_i} + \chi_{r_i} + \chi_{c_{i-1}}$

Proof. Notice that if I is an order ideal of $\mathsf{V} \times [k]$ and $\phi(I) = (\ell, c, r)$, then

$$(f_{i+1} - f_i)(I) = \begin{cases} 1 \text{ if } \ell = i+1 \text{ and } r \neq i+1, \\ 1 \text{ if } \ell \neq i+1 \text{ and } r = i+1, \\ 1 \text{ if } c = i \\ 2 \text{ if } \ell = r = i+1, \\ 0 \text{ otherwise.} \end{cases}$$

So $(f_{i+1} - f_i)$ counts the number of times $i+1$ appears in the left or right column of the orbit board, and the number of times i appears in the center column of the orbit board. We split this proof into two cases

Case 1. If a snake has a block labeled $i+1$ in the left or right column of the orbit board, then it won't have a block labeled i in the center column of the orbit and vise-versa. Therefore, if there are no two-tailed snakes, then by Lemma 4.2.8 there are six unique snakes and thus the sum of $(f_{i+1} - f_i)$ over the orbit is $\frac{6}{2(k+2)} = \frac{3}{k+2}$.

Case 2. Similarly, if there are two-tailed snakes, then by Lemma 4.2.8 there are two unique snakes, one with a block labeled $i+1$ in the left and right column twice and one block with label i in the center; and thus, the sum of $(f_{i+1} - f_i)$ over the orbit is $\frac{3}{k+2}$ ☐

This lemma can be generalized to differences between any two flux capacitors, not just successive ones.

Theorem 4.3.2. *For $k > 1$. Let $f_i = \chi_{\ell_i} + \chi_{r_i} + \chi_{c_{i-1}}$. The difference $f_i - f_j$ is $\frac{3(i-j)}{k+2}$-mesic under the action of rowmotion on order ideals of $\mathcal{J}(\mathsf{V} \times [k])$.*

Proof. First assume $i > j$. By telescoping sum we have

$$f_i - f_j = f_i - f_{i-1} + f_{i+1} - \cdots + f_{j+1} - f_j.$$

Grouping consecutive pairs and using Lemma 4.3.1, we get the average of $f_i - f_j$ is

$$\frac{3(i-j)}{k+2}.$$

Now suppose $j > i$. A similar argument shows the average of $f_j - f_i$ over an orbit is $\frac{3(j-i)}{k+2}$ so

$$\sum_{I \in \mathcal{O}} f_i - f_j = -\left(\sum_{I \in \mathcal{O}} f_j - f_i\right) = -\left(\frac{3(j-i)}{k+2}\right) = \frac{3(i-j)}{k+2}.$$

☐

Chapter 5

Proper colorings of a graph

5.1 Path graph

The action of whirling can result in interesting actions in settings other than order-reversing maps of posets. Let $G = (V, E)$ be a finite graph with $\#V = n$. A function $\kappa : V \to [m]$ is called an *m-coloring* of G. The coloring is *proper* if $\kappa(u) \neq \kappa(v)$ whenever $(u, v) \in E$. Let $\mathcal{F}_m(G)$ be set of all m-colorings of G and $\mathcal{K}_m(G)$ be the subset of all proper m-colorings of G. At the end of this section we will show a connection to toggling independent sets of a graph Section 5.3.

Example 5.1.1. Recall that $\mathcal{P}_n$ is the path graph with vertex set $[n]$ and edge set $\{\{i, i{+}1\} : i \in [n{-}1]\}$. Here is a proper 4-coloring of $\mathcal{P}_6$, $2 — 1 — 2 — 4 — 2 — 1 \in \mathcal{K}_4(\mathcal{P}_6)$. To stay consistent with previous sections, we will use the compact notation $212421 \in \mathcal{K}_4(\mathcal{P}_6)$.

In this section for a graph $G = (V, E)$ where $\#V = n$, we denote the whirl on

$\mathcal{K}_m(G)$ by $w = w_n w_{n-1} \dots w_1$ whirling at each vertex from left to right along the path.

Definition 5.1.2. Let $G = (V, E)$ be a graph. We define for each $v \in G$ the *color-characteristic* function $\chi_{v,i} : \mathcal{K}_m(G) \to \{0, 1\}$ as follows:

$$\chi_{v,i}(\kappa) = \begin{cases} 1 & \text{if } \kappa(v) = i, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Example 5.1.3. Consider the proper coloring $1323121 \in \mathcal{K}_3(\mathcal{P}_7)$. Note that $\chi_{1,1} = \chi_{5,1} = \chi_{7,1} = 1$. Here is an orbit of whirling on $\mathcal{K}_3(\mathcal{P}_7)$ containing 1323121.

$$\begin{matrix} 1 & 3 & 2 & 3 & 1 & 2 & 1 \\ 2 & 1 & 2 & 3 & 1 & 3 & 2 \\ 3 & 1 & 2 & 3 & 2 & 1 & 3 \\ 2 & 3 & 1 & 3 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 & 2 & 3 & 1 \\ 3 & 2 & 1 & 3 & 1 & 2 & 3 \end{matrix}$$

This action generalizes toggling independent sets of a path graph where now our graph is partitioned into independent sets, one for each color. When $m = 3$, get an extension of Theorem 1.2.8.

Theorem 5.1.4. *Fix any color $j \in [3]$. Set $\chi_i := \chi_{i,j}$. If w is whirling from left to right, then under the action of w on $\mathcal{K}_3(\mathcal{P}_n)$,*

1. *$\chi_i - \chi_{n+1-i}$ is 0-mesic, and*

2. *$2\chi_1 + \chi_2$ is 1-mesic and $\chi_{n-1} + 2\chi_n$ is 1-mesic.*

Observe in Example 5.1.3 that the number of 1's form the palindrome (2,2,3,0,3,2,2). Also the 2's form the palindrome (2,2,3,0,3,2,2), and the 3's form the palindrome (2,2,0,6,0,2,2).

Definition 5.1.5. Let $\kappa \in \mathcal{K}_m(\mathcal{P}_n)$. Define the *difference vector* of κ to be $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$, where $\alpha_i = \kappa(i) - \kappa(i+1) \in \mathbb{Z}/m\mathbb{Z}$.

In the case where $m = 3$, the difference vector only takes two values, either $+1$ or -1, which we represent by $\boldsymbol{+}$ and $\boldsymbol{-}$ respectively.

Example 5.1.6. The difference vector of $1323212 \in \mathcal{P}_7$ is $(\boldsymbol{-}, \boldsymbol{-}, \boldsymbol{+}, \boldsymbol{-}, \boldsymbol{-}, \boldsymbol{+})$.

As one might expect, the difference vector is not unique to a coloring. However if the color of any one vertex is known, the original coloring can be recovered uniquely. Therefore, there are exactly m different colorings for each difference vector.

Lemma 5.1.7. *Let $\kappa \in \mathcal{K}_3(\mathcal{P}_n)$ with difference vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$. Let w_i be the whirl at position i and α' be the difference vector of $w_i(\kappa)$. We get the following:*

1. *If $i = 1$, then $\alpha' = (-\alpha_1, \alpha_2, \ldots, \alpha_n)$. If $i = n$, then $\alpha' = (\alpha_1, \alpha_2, \ldots, -\alpha_n)$.*

2. *If $1 < i < n$, then $\alpha' = (\alpha_1, \ldots, \alpha_{i-2}, \alpha_i, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)$ (the positions of α_{i-1} and α_i interchange.)*

Proof. For (1), we will show what happens for $\kappa(1) = 2$. Since we are only whirling at the first position, $\alpha(i) = \alpha'(i)$ for $2 \leq i < n$, so we only need to check if $\alpha(1) = -\alpha'(1)$. There are two possibilities if $\kappa(1) = 2$:

1. If $\kappa = 21\ldots$, then $\alpha = (\boldsymbol{-}, \ldots)$, $w_1(\kappa) = 31\ldots$, and $\alpha' = (\boldsymbol{+}, \ldots)$.

2. If $\kappa = 23\ldots$, then $\alpha = (\textbf{+}, \ldots)$, $w_1(\kappa) = 13\ldots$, and $\alpha' = (\textbf{−}, \ldots)$.

For (2), we will show what happens when $\kappa(i) = 2$ for $1 < i < n$. We get the following:

1. If $\kappa = \ldots 121 \ldots$, then $\alpha = (\ldots, \textbf{+}, \textbf{−}, \ldots)$, $w_i(\kappa) = \ldots 131 \ldots$, and $\alpha' = (\ldots, \textbf{−}, \textbf{+}, \ldots)$.

2. If $\kappa = \ldots 323 \ldots$, then $\alpha = (\ldots, \textbf{−}, \textbf{+}, \ldots)$, $w_i(\kappa) = \ldots 313 \ldots$, and $\alpha' = (\ldots, \textbf{+}, \textbf{−}, \ldots)$.

3. If $\kappa = \ldots 123 \ldots$, then $\alpha = (\ldots, \textbf{+}, \textbf{+}, \ldots)$, $w_i(\kappa) = \ldots 123 \ldots$, and $\alpha' = (\ldots, \textbf{+}, \textbf{+}, \ldots)$.

4. If $\kappa = \ldots 321 \ldots$, then $\alpha = (\ldots, \textbf{−}, \textbf{−}, \ldots)$, $w_i(\kappa) = \ldots 131 \ldots$, and $\alpha' = (\ldots, \textbf{−}, \textbf{−}, \ldots)$.

All other cases follow in a similar manner. □

Example 5.1.8. In this example we will whirl in slow motion, taking note of the effects of each w_i on the difference vector. The red color will highlight the $\textbf{+} \leftrightarrow \textbf{−}$ flips and the blue color will highlight the adjacent swaps. This will motivate the next proposition.

$$
\begin{array}{lcccccclccccc}
 & 1 & 2 & 1 & 2 & 3 & 1 & & + & - & + & + & + \\
w_1 & 3 & 2 & 1 & 2 & 3 & 1 & & - & - & + & + & + \\
w_2 & 3 & 2 & 1 & 2 & 3 & 1 & & - & - & + & + & + \\
w_3 & 3 & 2 & 3 & 2 & 3 & 1 & & - & + & - & + & + \\
w_4 & 3 & 2 & 3 & 1 & 3 & 1 & & - & + & + & - & + \\
w_5 & 3 & 2 & 3 & 1 & 2 & 1 & & - & + & + & + & - \\
w_6 & 3 & 2 & 3 & 1 & 2 & 3 & & - & + & + & + & +
\end{array}
$$

Proposition 5.1.9. *Let $\kappa \in \mathcal{K}_3(\mathcal{P}_n)$, and let c be the leftward-cyclic shift map. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ is the difference vector of κ, then $c(\alpha)$ is the difference vector of $w(\alpha)$. Furthermore, the order of c on α divides the order of w on κ.*

Proof. As seen in part (2) of Lemma 5.1.7, the effect of w_i where i is in the interior of the path is interchanging values in the difference vector. Going from left to right this effectively moves the first entry of the difference vector to the end. Combine this with part (1) of Lemma 5.1.7, where the first entry (of the difference vector) has **+** and **−** swapped, and the same for final entry ☐

Example 5.1.10. Here is an orbit of proper-coloring whirling on $\mathcal{K}_3(\mathcal{P}_4)$ containing

1232. The corresponding difference vector to each proper coloring is to the right.

1	2	3	2		**+**	**+**	**−**
3	1	3	1		**+**	**−**	**+**
2	1	2	3		**−**	**+**	**+**
3	1	2	1		**+**	**+**	**−**
2	3	2	3		**+**	**−**	**+**
1	3	1	2		**−**	**+**	**+**
2	3	1	3		**+**	**+**	**−**
1	2	1	2		**+**	**−**	**+**
3	2	3	1		**−**	**+**	**+**

Notice that we ended up getting three orbits of c for one orbit of w. In fact, it turns out this will happen whenever $\sum_{i=1}^{n-1} \alpha_i$ does not divide 3.

Proposition 5.1.11. *Let $\kappa \in \mathcal{K}_3(\mathcal{P}_n)$ with difference vector α. If $\sum_{i=1}^{n-1} \alpha_i \equiv 0 \pmod 3$, then the order of w on κ divides $n-1$. If $\sum_{i=1}^{n-1} \alpha_i \not\equiv 0 \pmod 3$, then the order of w on κ divides $3(n-1)$.*

Proof. Assume $\sum_{i=1}^{n-1} \alpha_i \equiv 0 \pmod 3$. If $\alpha_1 = \boldsymbol{+}$, then $\kappa(2) = \kappa(1) + 1 \pmod 3$, so it must be $w\kappa(1) = \kappa(1) - 1 \pmod 3$. On the other hand, if $\alpha_1 = \boldsymbol{-}$, then $\kappa(2) = \kappa(1) - 1 \pmod 3$, so we know that $w\kappa(1) = \kappa(1) + 1 \pmod 3$. In either case we see $w\kappa(1) = \kappa(1) - \alpha_1 \pmod 3$. Using Proposition 5.1.9 we can extend this to $w^j\kappa(1) = \kappa(1) - (\alpha_1 + \cdots + \alpha_j) \pmod 3$. Therefore,

$$w^{n-1}\kappa(1) = \kappa(1) - (\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}) \pmod 3 = \kappa(1).$$

Since the order of c on α divides $n-1$ and $w^k(\kappa)(1) = \kappa(1)$, we know $w^k(\kappa) = \kappa$.

From Proposition 5.1.9 we know $n-1$ divides the order of w on κ, so the first result is obtained.

Now assume $\sum_{i=1}^{n-1} \alpha_i \not\equiv 0 \pmod 3$. Certainly $w^{3(n-1)}(\kappa) = \kappa$ from the arguments above so the order of w on κ divides $3(n-1)$ but does not divide $(n-1)$. We also know $(n-1)$ divides the order of w on κ from Proposition 5.1.9, so the order of w on κ is $3(n-1)$. $\square$

Proof of Theorem 5.1.4. Fix $\kappa \in \mathcal{K}_3(\mathcal{P}_n)$ with difference vector α, and let k be the order of c on α. If the order of w on κ is $3k$, then the orbit contains the other two proper colorings with difference vector α, therefore every color appears in each spot exactly $1/3$ of the time.

Now assume the order of w is k and fix some $i \in [n]$. We will show $\kappa(i) = w^{i-1}\kappa(n+1-i)$. From the previous proof we know

$$w^{r-1}\kappa(1) = \kappa(1) - \sum_{j=1}^{r-1} \alpha_j.$$

It follows that

$$w^{i-1}\kappa(n+1-i) = \kappa(1) - \sum_{j=1}^{i-1} \alpha_j + \sum_{j=i}^{n-1} \alpha_j \pmod 3.$$

However by rearranging we obtain

$$\begin{aligned}\sum_{j=1}^{i-1} -\alpha_j + \sum_{j=i}^{n-1} \alpha_j &= \sum_{j=1}^{i-1} -\alpha_j + \sum_{j=1}^{i-1} -\alpha_j + \sum_{j=1}^{i-1} \alpha_j + \sum_{j=i}^{n-1} \alpha_j \\ &= 2\sum_{j=1}^{i-1} -\alpha_j + \sum_{j=1}^{n-1} \alpha_j\end{aligned}$$

Since $\sum_{j=1}^{n-1} \alpha_j \equiv 0 \pmod 3$ and $2\sum_{j=1}^{i-1} -\alpha_j \equiv \sum_{j=1}^{i-1} \alpha_j \pmod 3$, we conclude that

$$w^{i-1}(\kappa)(n+1-i) = \kappa(1) + \sum_{j=1}^{i-1} \alpha_j = \kappa(i).$$

□

Example 5.1.12. Consider the orbit containing $1323121 \in \mathcal{K}_3(\mathcal{P}_7)$. Using the method from the proof of Theorem 5.1.4, we will match the 1 at $\kappa(5)$ with a 1 at $\kappa'(3)$ for some other proper coloring κ' in the orbit. In this example $i = 5$ so the matching 1 will be at $w^{i-1}\kappa(3) = w^4\kappa(3)$.

1	3	2	3	1	2	1
2	1	2	3	1	3	2
3	1	2	3	2	1	3
2	3	1	3	2	1	2
1	2	1	3	2	3	1
3	2	1	3	1	2	3

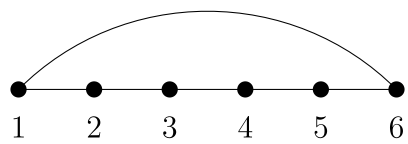


FIGURE 5.1: $\mathcal{C}_6$

5.2 Cycle graph

We get related results for the cycle graph, $\mathcal{C}_n$, where we connect the first and last vertices of $\mathcal{P}_n$. Taking a path graph and adding an edge between the first and last vertex results in a cycle graph.

Definition 5.2.1. We define $\mathcal{C}_n$ to be the *cycle graph* with vertex set $[n]$ and edge set $\{\{i, i+1\} : i \in [n-1]\} \cup \{n, 1\}$ as in Figure 5.1.

Theorem 5.2.2. *Fix any color $j \in [3]$. Set $\chi_i := \chi_{i,j}$. If $w = w_n \cdots w_2 w_1$ (whirling counterclockwise when the arc is drawn above), then under the action of w on $\mathcal{K}_3(\mathcal{C}_n)$,*

1. *The statistic $\chi_{3a+p} - \chi_{3b+p}$ is 0-mesic for $p \in [3]$ and $0 \leq a, b \leq \frac{n}{3} - 1$.*

2. *Furthermore, if $3 \not| n$, then χ_i is 1/3-mesic.*

Example 5.2.3. Here is the orbit of w containing $131232 \in \mathcal{K}_3(\mathcal{C}_6)$.

$$\begin{array}{cccccc}
1 & 3 & 1 & 2 & 3 & 2 \\
1 & 2 & 3 & 1 & 3 & 2 \\
3 & 1 & 2 & 1 & 3 & 1 \\
2 & 3 & 2 & 1 & 2 & 3 \\
1 & 3 & 2 & 3 & 1 & 2
\end{array}$$

Notice that the number of 1's in the first column match the number of 1's in the fourth column in accordance with Theorem 5.2.2.

To prove this theorem we will once again use the difference vector which we define slightly differently than for the path graph.

Definition 5.2.4. Let $\kappa \in \mathcal{K}_m(\mathcal{C}_n)$. Define the *difference vector* of κ to be

$$\alpha = (\alpha_1, \alpha_2 \ldots, \alpha_{n-1}, \alpha_n),$$

and where $\alpha_i = \kappa(i) - \kappa(i+1) \in \mathbb{Z}/m\mathbb{Z}$ where $\kappa(n+1) = \kappa(1)$.

Again since our goal is to prove a theorem where the number of colors is 3, our difference vectors are composed of $+1$'s and -1's, which we represent with **+** and **−** respectively

Example 5.2.5. The proper 3-coloring $1323 \in \mathcal{K}_3(\mathcal{C}_4)$ has difference vector (**−**, **−**, **+**, **+**).

Lemma 5.2.6. *If $\kappa \in \mathcal{K}_3(\mathcal{C}_n)$ and α is the difference vector of κ, then $\sum_{i=1}^{n} \alpha_n = 0 \pmod 3$.*

Proof. Since $\kappa(j) = \kappa(1) + \sum_{i=1}^{j-1} \alpha_i \pmod 3$, we have $\kappa(1) + \sum_{i=1}^{n} \alpha_i = \kappa(1) \pmod 3$. □

Definition 5.2.7. We define *skip-leftward cyclic shift*, denoted by $\widehat{c}$, as follows:

$$\widehat{c}(\alpha_1, \alpha_2, \ldots \alpha_n) = (\alpha_2, \ldots, \alpha_1, \alpha_n).$$

Lemma 5.2.8. *Let $\kappa \in \mathcal{K}_m(\mathcal{C}_n)$, where the indices now work mod n. If α is the difference vector of κ and α' is the difference vector of $w\kappa$, then $\alpha' = \widehat{c}(\alpha)$.*

Proof. Let α be the difference vector of $\kappa \in \mathcal{K}_3(\mathcal{C}_n)$. Since every vertex is an interior vertex, we know that whirling at i results in a swap between α_{i-1} and α_i in α by Lemma 5.1.7. When applying w, first positions n and 1 are swapped, then 1 and 2, and so on until $n-1$ and n swap. This last interchange returns α_n to its original location, leaving α_1 in the penultimate position. Every other α_i has shifted leftward one spot. $\square$

Example 5.2.9. Let α be the difference vector of $\kappa \in \mathcal{K}_3(\mathcal{C}_4)$ and α' the difference vector of $w\kappa$. If $\kappa = 1323$, then $\alpha = (\boldsymbol{-}, \boldsymbol{-}, \boldsymbol{+}, \boldsymbol{+})$, $w\kappa = 2121$, and $\alpha' = \widehat{c}(\alpha) = (\boldsymbol{-}, \boldsymbol{+}, \boldsymbol{-}, \boldsymbol{+})$.

Using just these lemmas, we can prove the second statement of Theorem 5.2.2.

Proof of Theorem 5.2.2.2. Assume that $3 \nmid n$, $\kappa \in \mathcal{K}_3(\mathcal{C}_n)$ with difference vector α and without loss of generality that $\kappa(1) = 1$ and $\alpha_n = \boldsymbol{+}$. By Lemma 5.2.8 we know after $n-1$ applications of w the difference vector will be α again. If we can show $w^{n-1}(\kappa)(1) \neq \kappa(1)$, then the orbit contains the other two proper colorings with difference vector α, therefore every color appears in the each spot exactly 1/3 of the time.

The color of the first vertex will only change when $\alpha_n \neq \alpha_1$, which is when $\alpha_1 = \boldsymbol{-}$. Therefore, the color in in the first position will increment by one for each $\boldsymbol{-}$ in the difference vector. Similarly if $\alpha_n = \boldsymbol{-}$, then the color of the first vertex will decrement by one for each $\boldsymbol{+}$ in the difference vector, which is the same as increasing by the number of $\boldsymbol{-}$'s since $\sum_{i=1}^{n} \alpha_i = 0$. So the proof is complete as long as the number of $\boldsymbol{-}$'s in the difference vector is not a multiple of 3. However, if the number of $\boldsymbol{-}$'s in the difference vector is a multiple of 3, then so is the number of $\boldsymbol{+}$'s in the difference vector by Lemma 5.2.6. But n is the sum of the number of $\boldsymbol{+}$'s and $\boldsymbol{-}$'s in the difference vector and that $3 \nmid n$. $\square$

To prove the first part of Theorem 5.2.2 we need the following lemma.

Lemma 5.2.10. *If $0 \leq a \leq \frac{n}{3} - 1$ and $p \in [3]$, then $w^{3a}\kappa(p) = \kappa(3a + p)$. In words, the colorings in columns p and column $3a + p$ are the same in an orbit board.*

Proof. Let $L = \alpha_1 + \alpha_2 + \cdots + \alpha_{3a}$ and let $P = \alpha_{3a+1} + \cdots + \alpha_{3a+p-1}$. Without loss of generality, we may assume $\alpha_n = \boldsymbol{+}$. We know from the definition of difference vectors,

$$\kappa(3a + 1) = \kappa(1) + L \pmod 3 \text{ and } \kappa(p + 3a) = \kappa(1) + P + L \pmod 3.$$

Let $L_{\boldsymbol{-}} = \#\boldsymbol{-} \in \{\alpha_1, \alpha_2, \ldots, \alpha_{3a}\}$. Since L is the sum of $3a$ terms, we know $L = L_{\boldsymbol{-}} \pmod 3$. It follows from the argument from the previous proof that $w^{3a}\kappa(1) = \kappa(1) + L_{\boldsymbol{-}}$, or rather,

$$w^{3a}\kappa(1) = \kappa(1) + L \pmod 3.$$

Add P to both sides to get

$$w^{3a}\kappa(1) + P = \kappa(p + 3a) \pmod 3.$$

Notice that adding P on the left-hand side is the same as moving p positions to the right in the orbit board so we have,

$$w^{3a}\kappa(p) = \kappa(p + 3a) \pmod 3.$$

□

Example 5.2.11. Here is the orbit of w containing $131232 \in \mathcal{K}_3(\mathcal{C}_6)$. The difference vector is $(\boldsymbol{-}, \boldsymbol{+}, \boldsymbol{+}, \boldsymbol{+}, \boldsymbol{-}, \boldsymbol{-})$.

$$\begin{array}{cccccc} 1 & 3 & 1 & 2 & 3 & 2 \\ 1 & 2 & 3 & 1 & 3 & 2 \\ 3 & 1 & 2 & 1 & 3 & 1 \\ 2 & 3 & 2 & 1 & 2 & 3 \\ 1 & 3 & 2 & 3 & 1 & 2 \end{array}$$

The previous lemma outlines the following bijection between the circled 3's in the orbit board above. Let $L = \alpha_1 + \alpha_2 + \alpha_3$ and $P = \alpha_4$. Since $\alpha_n = \boldsymbol{-}$ we have $w^3\kappa(1) = \kappa(1) - L_{\boldsymbol{+}} \pmod 3 = 2$. But since L is length 3, we know $L = -L_{\boldsymbol{+}} \pmod 3$, so $w^3\kappa(1) = \kappa(4)$. Finally, add P to get $w^3\kappa(2) = \kappa(5)$.

Proof of Theorem 5.2.2.1. Lemma 5.2.10 gives us $\chi_i - \chi_{3a+i}$ is 0-mesic for all $0 \leq a \leq \frac{n}{3} - 1$ and $i \in [3]$. Through linear combinations we may obtain $\chi_{3a+i} - \chi_{3b+i}$ is 0-mesic for any $0 \leq a, b \leq \frac{n}{3} - 1$ and $i \in [3]$. □

The reader should notice that nowhere did we use the fact that $3 \mid n$.

5.3 Partial proper colorings

Definition 5.3.1. Similar to a (proper) coloring; we say a *partial proper coloring* of a graph $G = (V, E)$ is a map $\pi : V \to [0, n]$ such that if $\pi(x) = \pi(y)$, then one of the following must be true,

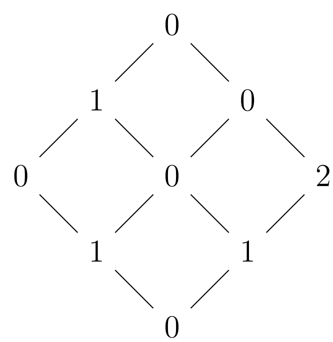
1. $x = y$,

2. $(x, y) \notin E$,

3. or $\pi(x) = \pi(y) = 0$.

The first two conditions are what give us proper colorings, the addition of the third lets us have vertices labeled 0 be next to each other. We will call vertices labeled with 0 *uncolored*, with the remaining labels forming a proper coloring the other vertices.

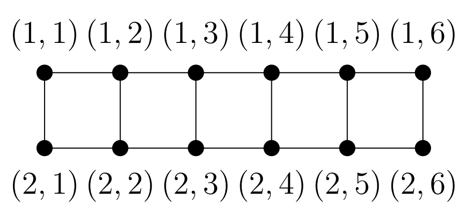
Example 5.3.2. Here is a partial proper 2-coloring of $G = [3] \times [3]$:



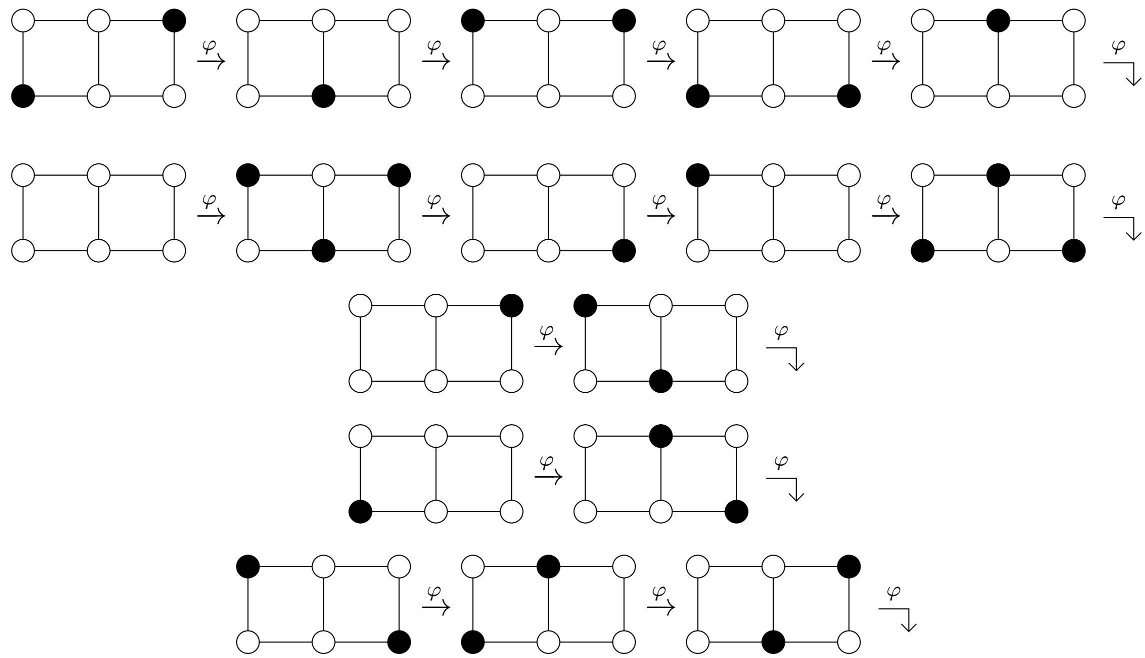
When $\pi : V \to [0, 1]$, a partial proper coloring is precisely an independent set of G, so whirling on these generalizes toggling independent sets of G (cf. Section 1.2)

Definition 5.3.3. Define $R_n = [2] \times [n]$ to be the product of a path graph of length 2 and one of length n. The vertices of R_n are pairs (i, j) where $i \in [2]$ and $j \in [n]$.

Example 5.3.4. Here is R_6 with the nodes labeled.



Example 5.3.5. The orbits of $\mathcal{I}(R_3)$ under $\varphi = \tau_{(2,n)}\tau_{(1,n)}\cdots\tau_{(2,1)}\tau_{(1,1)}$ where τ_v was defined in Definition 1.2.3.

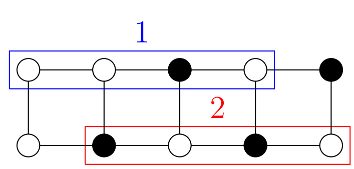


Conjecture 5.3.6. *Fix $n \geq 2$ and let $\varphi = \tau_{(2,n)}\tau_{(1,n)}\cdots\tau_{(2,1)}\tau_{(1,1)}$ Coxeter toggling on $\mathcal{I}(R_n)$, then*

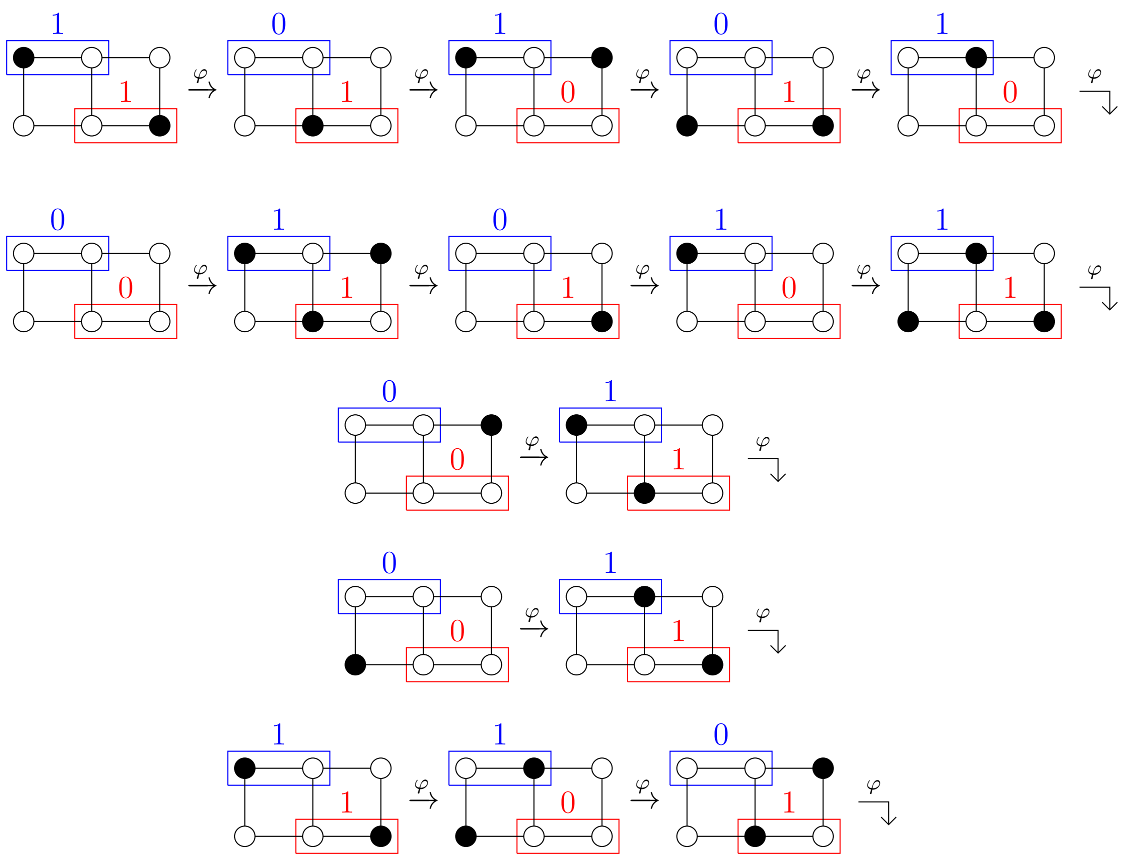
$$\sum_{i=1}^{n-1} \chi_{(1,i)} - \sum_{i=2}^{n} \chi_{(2,i)}$$

is 0-mesic, that is, the total number of times $(1,1), \ldots, (1, n-2)$, $(1, n-1)$ appears in a set S in an orbit of φ is the same as the number of times $(2,2), \ldots, (2, n-1)$, $(2, n)$ appears.

We can imagine putting two single row boxes that exclude opposite corners



and counting the number of times a vertex is colored black across an orbit is in those boxes. Here is an example on R_3.

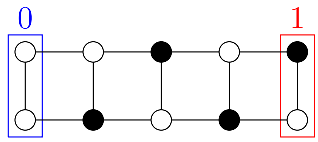


Proposition 5.3.7. *Fix $n \geq 2$ and let $\varphi = \tau_{(2,n)}\tau_{(1,n)}\cdots\tau_{(2,1)}\tau_{(1,1)}$ toggling on $\mathcal{I}(R_n)$, then*

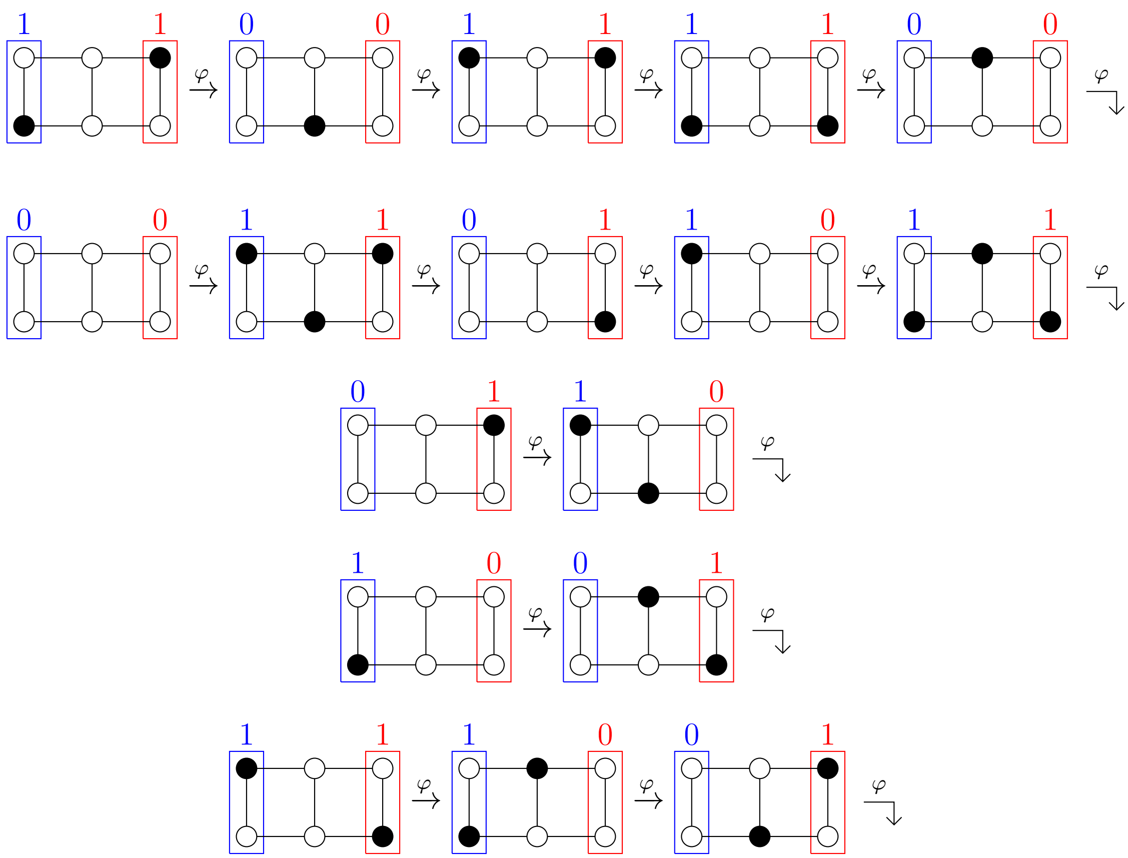
$$\chi_{(1,1)} + \chi_{(2,1)} - (\chi_{(1,n)} + \chi_{(2,n)})$$

is 0-mesic, that is, the number of times $(1,1)$ or $(2,1)$ appears in an independent set S in an orbit of φ is the same as the number of times $(1,n)$ or $(2,n)$ appears in an independent set S in an orbit of φ.

Again, we can imagine putting two single column boxes over the first and last column of R_n



and counting the number of times a vertex is in those boxes. Here is an example on R_3.

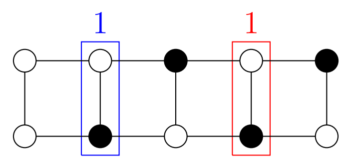


Conjecture 5.3.8. *For $\mathcal{I}(R_n)$ with $n \geq 2$ given and let $\varphi = \tau_{(2,n)}\tau_{(1,n)} \cdots \tau_{(2,1)}\tau_{(1,1)}$, then*

$$\chi_{(1,2)} + \chi_{(2,2)} - (\chi_{(1,n-1)} + \chi_{(2,n-1)})$$

is 0-mesic, that is, the number of times $(1,2)$ or $(2,2)$ appears in an independent set S in an orbit of φ is the same as the number of times $(1,n-1)$ or $(2,n-1)$ appears *in an independent set S in an orbit of φ.*

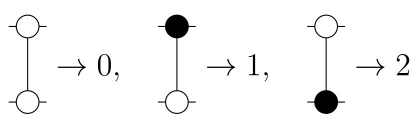
Again, we can imagine putting two single column boxes over the second and penultimate column of R_n



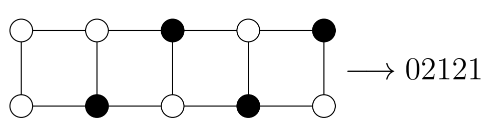
and counting the number of times a vertex is in those boxes. Displaying a set of pictures like this for R_4 would take up more than a page in length. With more than enough motivation, here we connect these independent sets of R_n to partial proper colorings of $\mathcal{P}_n$ via an equivariant bijection that sends coxeter toggling to whirling.

Definition 5.3.9. Let $\mathcal{T}_k(\mathcal{P}_n)$ be the set of function $f : [n] \to [0, k]$ such that when written as a word, f is jj avoiding for all $j \in [k]$.

There is a bijection between $\mathcal{I}(R_n)$ and $T_2(\mathcal{P}_n)$ given by the following map. For $S \in \mathcal{I}(R_n)$ we map the columns of R_n as follows:



Example 5.3.10.



Let K_m be the complete graph with vertex set $[m]$. Define $\mathcal{Q}_{m\times n} = K_m \times [n]$, the product of K_m and a path graph of length n. Since $K_2 = [2]$, $\mathcal{Q}_{2\times n} = R_n$. We define $\varphi_i = \tau_{(m,i)} \cdots \tau_{(1,i)}$ on $\mathcal{Q}_{m\times n}$ and $\varphi = \varphi_n \cdots \varphi_1$.

Lemma 5.3.11. *There is an equivariant bijection between $\mathcal{I}(\mathcal{Q}_{m\times n})$ and $T_m(\mathcal{P}_n)$ that sends φ to w.*

Proof. First we will consider the case where $m = 2$. The bijection is as written above where the empty columns of R_n gets mapped to 0, a column with a vertex included in the first row gets mapped to 1, and a column with a vertex included in the second row gets mapped to 2. Let $S \in \mathcal{I}(R_n)$. Let v_1 and v_2 be two vertices in the i'th column. Notice that toggling the first row in the i'th column and then the second row will perform the following action in the i'th column: If v_1 is in the set, then remove v_1 and add v_2 if possible, otherwise leave the column empty. If v_2 is in the set, then remove v_2, leaving the column empty. If the column is empty, then add v_1 if possible, if not, then add v_2 if possible, if not, leave the column empty. Via correspondence this definition matches with whirling, w_i.

The case where $m > 2$ is similar. Let $m > 2$ and $S \in \mathcal{I}(\mathcal{Q}_{m\times n})$. We will construct $\pi \in T_m(\mathcal{P}_n)$ to establish a bijection between $\mathcal{Q}_{m\times n}$ and $\pi \in T_m(\mathcal{P}_n)$. The vertices $\{(1, j), \ldots , (m, j)\}$ form a complete graph in $\mathcal{Q}_{m\times n}$, call this set K_m^j. If K_m^j does not intersect with S, then set $\pi(j) = 0$. Otherwise, there is at most one vertex $(i, j) \in K_m^j$ such that $(i, j) \in S$, so set $\pi(j) = i$. With this bijection we will now check that φ gets sent to w. It suffices to show that φ_j is sent to w_j. The action of φ_j toggles at each vertex of K_m^j once from $(1, j)$ to (m, j). The effect of this is that the lowest possible $i > j$ or $i = 0$ will be in $\varphi_j(S)$. However, this is exactly the action of w_j. □

Example 5.3.12. Given $02121 \in T_2(\mathcal{P}_n)$ we see

$$02121 \xrightarrow{w} 10002.$$

We can now write orbits more compactly as orbit boards of partial proper colorings. Here are the nine orbits of $T_2(\mathcal{P}_4)$.

$$\begin{array}{cccc} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{cccc} 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 2 \\ 1 & 0 & 0 & 0 \end{array} \qquad \begin{array}{cccc} 0 & 1 & 2 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 \end{array} \qquad \begin{array}{cccc} 0 & 0 & 1 & 2 \\ 1 & 2 & 0 & 0 \end{array} \qquad \begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{cccc} 2 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{array} \qquad \begin{array}{cccc} 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \end{array} \qquad \begin{array}{cccc} 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \end{array} \qquad \begin{array}{cccc} 2 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{array}$$

Checking Proposition 5.3.7 can be reduced to counting the number of 0's in a column.

We will now prove Proposition 5.3.7. We will do so by drawing a connecting sequence of 0's between individual 0's in the left column of the orbit board and 0's of the right column of the orbit board. Here is an example from above with the zeroes highlighted.

$$\begin{array}{cccc} 2 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{array}$$

The following lemma describes how these connected zero strings appear in an orbit board of w.

Lemma 5.3.13. *Let $\mathcal{B}$ be an orbit board where $\mathcal{B}(i,j)$ is the jth element in the ith row of an orbit.*

1. *If $\mathcal{B}(i,0) = \mathbf{0}$, then $\mathcal{B}(i+1,0) \neq \mathbf{0}$. (A $\mathbf{0}$ in the first position is never above another $\mathbf{0}$.)*

2. *If $\mathcal{B}(i,n) = \mathbf{0}$, then $\mathcal{B}(i+1,n) \neq \mathbf{0}$. (A $\mathbf{0}$ in the last position is never above another $\mathbf{0}$.)*

3. *If $\mathcal{B}(i,j) = \mathcal{B}(i,j+1) = \mathbf{0}$, then $\mathcal{B}(i+1,j) \neq \mathbf{0}$. (If a zero is to the left of a zero, then there is not a zero below it.)*

4. *If $\mathcal{B}(i,j) = \mathcal{B}(i+1,j) = \mathbf{0}$, then $\mathcal{B}(i+2,j) \neq \mathbf{0}$. (Three zeros do not appear on top of each other.)*

5. *If $\mathcal{B}(i,j) = \mathbf{0}$, $\mathcal{B}(i+1,j) \neq \mathbf{0}$, and $\mathcal{B}(i,j+1) \neq \mathbf{0}$, then $\mathcal{B}(i+1,j+1) = \mathbf{0}$. (If a zero is not bordered by a zero, then there is one diagonally down right.)*

Proof. For (1), (2), and (3), the only way we can whirl at **0** and get a **0** is if the **0** is surrounded by a **1** and a **2**, since edge positions only have one neighbor this is impossible.

For (4), a zero can only be above another zero if its neighbors are **1** and **2**, which means the zero below it will have a zero neighbor. Similarly, if a zero has a nonzero neighbor, then it cannot be above a zero. Thus three zeros cannot be on top of each other.

For (5), Suppose a **2** is right of the **0**, then directly below the **2** must be **0**. If a **1** is to the right of the **0**, then what is below the **0** must be a **2** since toggling at the **0** cannot result in a **1**, thus whirling at the **1** results in **0**. □

From this proposition we know that these zero strings do not split or combine and thus there is a bijection between the zeros on the left and of the board and the zeros on the right of the board. Thus proving Proposition 5.3.7.

Proof of Proposition 5.3.7. Using Lemma 5.3.13, since each zero string has one zero in the first column of $\mathcal{B}$ and one zero in the last position $\mathcal{B}$ this proves the number of zeros in the first an last columns are the same for each orbit. Since the number of zeros in the first and last column are the same, so are the number of **1**'s and **2**'s in which the homomesy follows. □

For Conjecture 5.3.8 we cannot do something similar because in middle columns of an orbit board, two 0's can appear one directly above another. as in the last example. Instead we will need a different way to relate the 2nd and penultimate column.

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