Rowmotion on the chain of V's poset and whirling dynamics

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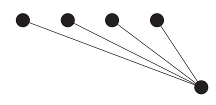
Overview

- We connect the well-studied operation of *rowmotion* on the order ideals of a finite poset with the less familiar *whirling* action on P-partitions with bounded labels.
- One of our main results is an equivariant bijection that carries one to the other for any finite poset P.
- We then leverage this to study the rowmotion action on the "chain of V's" poset $\mathsf{V}_k := \mathsf{V} \times [k]$ (a 3-element V-shaped poset cross a finite chain), which has surprisingly good dynamical properties.
- We also generalize this to the case where we replace V with a n-**claw**, a poset with a single minimal element covered by exactly n incomparable elements. In both cases we obtain both periodicity results and homomesy.



Definitions

Definition: Homomesy [PrR13]

Let G be a finite group acting on a finite set S. Let $\mathrm{st} : S \to \mathbb{Q}$ be a statistic. If $\mathcal{O} \subseteq S$, then we let

$$\mathrm{st}\,\mathcal{O} = \sum_{x \in \mathcal{O}} \mathrm{st}\,x.$$

Call st *homomesic* if $\mathrm{st}\,\mathcal{O}/\#\mathcal{O}$ is constant over all orbits $\mathcal{O}$, where the hash-tag means cardinality. In particular, st is c-*mesic* if, for all orbits $\mathcal{O}$,

$$\frac{\mathrm{st}\,\mathcal{O}}{\#\mathcal{O}} = c.$$

Homomesy for binary strings under rotation [PrR13]

$S_{n,k} := \{w_1 w_2 \dots w_n \mid w_i \in \{0,1\}$ for all i, and having k ones$\}$

under the cyclic rotation map, $w_1 w_2 \dots w_n \mapsto w_n w_1 \dots w_{n-1}$, with the *inversion statistic*

$$\mathrm{inv}\, w_1 w_2 \dots w_n = \#\{(i,j) \mid i < j \text{ and } w_i > w_j\}.$$

Ex. When $n = 4$ and $k = 2$ there are two orbits

w	inv w	w	inv w
1100	4	1010	3
0110	2	0101	1
0011	0		
1001	2		
average $= 8/4 = 2$		average $= 4/2 = 2$	

Definition: Rowmotion and transfer maps

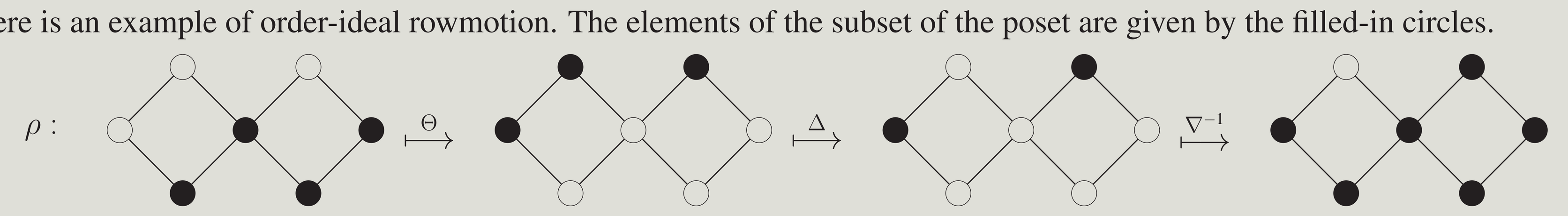
Let P be a generic poset. We define natural bijections between the sets $\mathcal{J}(P)$ of all *order ideals* (aka downsets) of P, $\mathcal{F}(P)$ of all *order filters* (aka upsets) of P, and $\mathcal{A}(P)$ of all *antichains* of P.

- The map $\Theta : 2^P \to 2^P$ where $\Theta(S) = P \setminus S$ is the **complement** of S (sending order ideals to filters and vice versa).
- The **up-transfer** $\nabla : \mathcal{J}(P) \to \mathcal{A}(P)$, where $\nabla(I)$ is the set of maximal elements of I. For an antichain $A \in \mathcal{A}(P)$, $\nabla^{-1}(A) = \{x \in P : x \le y$ for some $y \in A\}$.
- The **down-transfer** $\Delta : \mathcal{F}(P) \to \mathcal{A}(P)$, where $\Delta(F)$ is the set of minimal elements of F. For an antichain $A \in \mathcal{A}(P)$, $\Delta^{-1}(A) = \{x \in P : x \ge y$ for some $y \in A\}$.

Order-ideal rowmotion (shortened for this poster to just **rowmotion**) is the map $\rho : \mathcal{J}(P) \to \mathcal{J}(P)$ given by the composition $\rho_J = \nabla^{-1} \circ \Delta \circ \Theta$.

Example of Order Ideal Rowmotion

Here is an example of order-ideal rowmotion. The elements of the subset of the poset are given by the filled-in circles.



Definition: Rowmotion Toggles

A bijective function $f : P \to [p]$ (with $\#P = p$) such that $f(x) < f(y)$ whenever $x <_P y$ is called a *linear extension*. We denote by $\mathcal{L}(P)$ the set of *all linear extensions* of P. Rowmotion can be realized as a composition of toggling involutions by Cameron and Fon-der-Flaass [CF95] "toggling once at each element of P along any linear extension (from top to bottom)".

$$\widehat{\tau}_x(I) = \begin{cases} I \setminus \{x\} & \text{if } x \in I \text{ and } I \setminus \{x\} \in \mathcal{J}(P) \\ I \cup \{x\} & \text{if } x \notin I \text{ and } I \cup \{x\} \in \mathcal{J}(P) \\ I & \text{otherwise.} \end{cases}$$

Toggles are the $k = 1$ case of the more general definition of whirling below.

Definition: Whirling

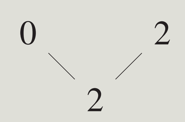
Let $[0,k] = \{0,1,2,\dots,k\}$ and $\mathcal{F} \subseteq [0,k]^X$, where X is a set, be a family of functions $f : X \to [0,k]$.
For $f \in \mathcal{F}$ we define the *whirl* $w_x : \mathcal{F} \to \mathcal{F}$ at $x \in X$ as follows: repeatedly add 1 (modulo $k+1$) to the value of $f(i)$ until we get a function in $\mathcal{F}$, leaving all other labels alone. As before whirling depends on the ambient family of functions $\mathcal{F}$. Then *whirling* $w : \mathcal{F} \to \mathcal{F}$ is the composition of all w_x for $x \in X$, taken in some fixed order.
Joseph, Propp, and Roby [JPR18], initially definied whirling on function families (such as injective or surjective) $\mathcal{F} \subseteq [k]^{[n]}$, with $w := w_n w_{n-1} \cdots w_1$, obtaining homomesy results and conjectures for the indicator function statistics. We generalize this to whirling k-bounded P-partitions of posets [PR24+].

Example of Whirling functions

Let $\mathcal{F} = \{f \in [0,4]^{[5]} : f(1) \ne f(2)\}$. If we apply w_2 to $f = 43213$, adding 1 in the second position gives 44213, but this is not in $\mathcal{F}$. Adding 1 again in this position gives the result: $w_2(f) = 40213$.

Definition: k-bounded P-Partitions

A P-*partition* is a map σ from P to $\mathbb{N}$ such that if $x <_P y$, then $\sigma(x) \ge \sigma(y)$ [Sta11].
A k-*bounded* P-*partition* is a function $f : P \to [0,k]$ such that if $x \le_P y$, then $f(x) \ge f(y)$. Let $\mathcal{F}_k(P)$ be the set of all such functions.



A k-bounded P-Partition on V.

Definition: Whirling k-bounded P-partitions

For any linear extension $(x_1, x_2, \dots, x_p) \in \mathcal{L}(P)$. We define *whirling* k-bounded P-partitions as $w = w_{x_1} w_{x_2} \cdots w_{x_p}$, iteratively computing the whirl at each poset elemnt down a linear extension. This is independent of the choice of linear extension since w_x and w_y commute whenever neither x covers y nor y covers x.

Equivariant bijection between order-ideal rowmotion on poset $P \times [k]$ and whirling P-partitions on P

There is a natural bijection between order ideals of a poset P and 1-bounded P-partitions in $\mathcal{F}_1(P)$. Specifically, a 1-bounded P-partition in $\mathcal{F}_1(P)$ is simply the indicator function of an order ideal $I \in \mathcal{J}(P)$. We extend this to an equivariant bijection $\mathcal{F}_k(P) \to \mathcal{J}(\mathcal{P} \times [k])$ which sends w to ρ.

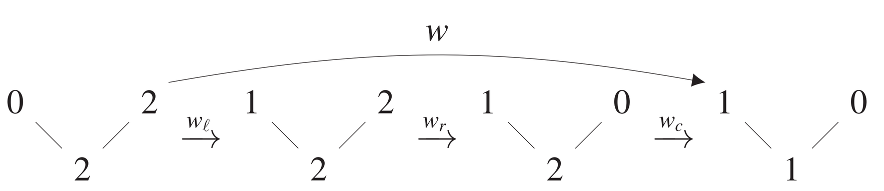
We call the chains $\{(x,1),(x,2),\dots,(x,k)\} \subseteq P \times [k]$, for $x \in P$, the *fibers* of $P \times [k]$, and construct an equivariant bijection that first sends w_x to order-ideal toggling down the fiber $\{(x,1),(x,2),\dots,(x,k)\}$.

Theorem [PR24+]: Connecting whirling with rowmotion

For any linear extension $(x_1, x_2, \dots, x_p) \in \mathcal{L}(P)$. There is an equivariant bijection between $\mathcal{F}_k(P)$ and $\mathcal{J}(P \times [k])$ which sends whirling, $w = w_{x_1} w_{x_2} \cdots w_{x_p}$, to rowmotion on $\mathcal{J}(P \times [k])$.

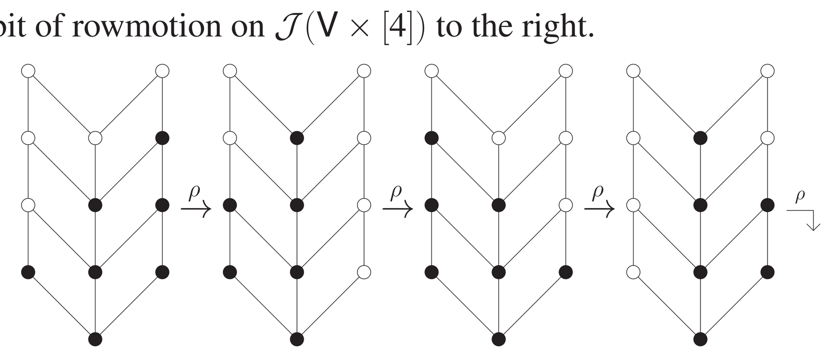
Whirling P-partitions of V and Claw posets

Let P be the V poset with labels ℓ, c, r, $k = 2$, and $w = w_c w_r w_\ell$.



Here is the orbit of whirling on $\mathcal{F}_4(\mathsf{V})$ corresponding to the orbit of rowmotion on $\mathcal{J}(\mathsf{V} \times [4])$ to the right.

$$(1,3,3) \xrightarrow{w} (2,4,0) \xrightarrow{w} (3,3,1) \xrightarrow{w} (0,4,2) \xrightarrow{w}$$



Whorm Anatomy

Orbit boards of whirling on $\mathsf{V} \times [k]$ can be uniquely partitioned into tiles called *whorms*. Whorms are maximal collections of labels in the orbit board, two labels on the row are connected if they have the same label and one is the label of the minimal element, and vertically connected if the label above is one less than the label below.

Whorms are composed of two parts. A *head* which lies in the column of the minimal element, and the *tail(s)* which lie in any of the other columns. In the case of $\mathsf{V} \times [k]$ we have either *one-tailed whorms*, or *two-tailed whorms*. We think of the whorms as starting with the label 0 and terminating with the label k. Let $\mathsf{t}(\varsigma)$ be the number of rows containing elements of the tail, and let $\mathsf{h}(\varsigma)$ be the number of rows containing elements of the head. It follows $\mathsf{t}(\varsigma) + \mathsf{h}(\varsigma) = k + 2$ for each whorm ς.

Since all whorms terminate in the column of the minimal element, they get a natural cyclic ordering. Two whorms are *consecutive* if one whorms head is adjacent to and above to the other. We number them moving down the poset. We will use the orbit board of $\mathsf{V} \times [4]$ on the right as an example; let ς_1 be the green whorm, ς_2 be the red whorm, ς_3 be the blue whorm, and ς_4 be the orange whorm. For example, $\mathsf{t}(\varsigma_1) = 4$ and $\mathsf{h}(\varsigma_1) = 2$.

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

The same orbit board of whirling k-bounded P-partitions of $\mathsf{V} \times [4]$ twice, the one on the left highlights a single whorm, the one on the right highlights all whorms.

Lemma [PR24+]

Given an orbit board of whirling k-bounded P-partitions of $\mathsf{V} \times [k]$ with one-tailed whorms, let ς_1, ς_2, ς_3, and ς_4 be consecutive, then

$$\mathsf{t}(\varsigma_4) = \mathsf{t}(\varsigma_1).$$

Furthermore, if the orbit board contains two-tailed whorms, then $\mathsf{t}(\varsigma_1) = \mathsf{t}(\varsigma_3)$.

Proposition [PR24+]

Given an orbit board of whirling k-bounded P-partitions of $\mathsf{V} \times [k]$ with one-tailed whorms, there are at most six distinct whorms.

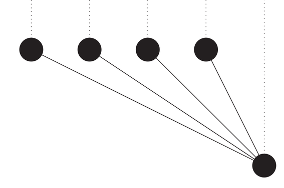
We extend our results for the chain of V's poset to a "chain of claws" poset, defined below. The equivariant bijection and techniques described already extend with only limited difficulty to this new setting.

The *claw poset*, C_n, has elements $\{b_1, \dots, b_n, \widehat{0}\}$ where each b_i covers $\widehat{0}$. For example, the Hasse diagram of C_4 would be . The *chain of claws poset* is defined to be $\mathsf{C}_n \times [k]$. Using the established equivariant bijection between $\mathcal{J}(\mathsf{C}_n \times [k])$ and k-bounded P-partitions $\mathcal{F}_k(\mathsf{C}_n)$ that sends rowmotion to whirling, we can prove similar homomesies and periodicity to that of the special case $\mathsf{C}_2 = \mathsf{V}$. Instead of *triples* of numbers, we will consider orbit boards of $(n+1)$-*tuples* on $[0,k]$, $\left(f(b_1), f(b_2), \dots, f(b_n), f(\widehat{0})\right)$, satisfying $f(b_i) \le f(\widehat{0})$ for each $i \in [n]$.

Note that the first and third columns in orbit board on the right are identical. This is no accident. If two (or more) entries among the first n in a given row are the *same* , then those positions (columns) remain the same throughout the *entire* orbit board. This is because the entries $b_1, \dots, b_n$ represent the result of whirling at *incomparable* elements of the poset C_n. Furthermore, these two entries must belong to the same whorm, because each will be whorm-connected via $\widehat{0}$ exactly when their values match with the value of the minimal element. So orbit boards for whirling on C_n can decompose into multi-tailed worms with up to n tails.

There are similar, but harder to state, propositions for the number of whorms in an orbit of whirling k-bounded P-partitions of C_n and their symmetries in the paper.

1	0	1	2	3
2	1	2	3	3
3	2	3	0	3
0	3	0	1	3
1	0	1	2	2
2	1	2	0	3
3	2	3	1	3
0	3	0	2	3
1	0	1	3	3
2	1	2	0	2
0	2	0	1	3
1	3	1	2	3
2	0	2	3	3
3	1	3	0	3
0	2	0	1	2



An orbit board of whirling on k-bounded P-partitions of $\mathsf{C}_4 \times [3]$. The claw poset C_4 is drawn underneath to indicate each column's associated element in C_4.

Homomesy and Order for rowmotion on order ideals of $\mathsf{V} \times [k]$

Theorem [PR24+]]: Order of rowmotion on a chain of V's

The order of rowmotion on $\mathcal{J}(\mathsf{V} \times [k])$ is $2(k+2)$. Furthermore, $\rho(I)$ is an order ideal which is a reflection of I across the center fiber.

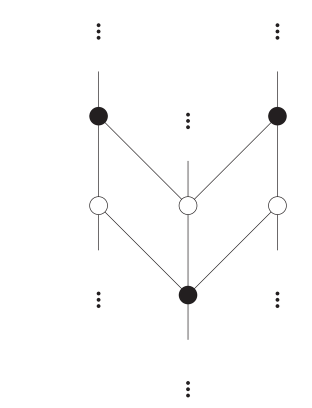
Theorem [PR24+]: Symmetry of statistics across the center chain

Let χ_x be the indicator function for $x \in \mathsf{V} \times [k]$. We have the following homomesies for the action of ρ on $\mathcal{J}(\mathsf{V} \times [k])$

1. The statistic $\chi_{\ell_i} - \chi_{r_i}$ is 0-mesic for all $i \in [k]$.
2. The statistic $\chi_{\ell_1} + \chi_{r_1} - \chi_{c_k}$ is $\frac{2(k-1)}{k+2}$-mesic.

Theorem [PR24+]: Homomesy of flux capacitor configuration of cardinality statistics

For $k > 1$. Let $F_i = \chi_{\ell_i} + \chi_{r_i} + \chi_{c_{i-1}}$ (we call this a configuration a "flux capacitor"). Under the action of rowmotion on order ideals of $\mathcal{J}(\mathsf{V}_k)$, the difference of arbitrary flux-capacitors is $F_i - F_j$ is $\frac{3(j-i)}{k+2}$-mesic.



A section of $\mathsf{V} \times [k]$ with elements of a flux capacitor configuration, ℓ_i, r_i and c_{i-1}, shaded black.

Homomesy and Order for rowmotion on order ideals of $\mathsf{C}_n \times [k]$

Theorem [PR24+]: Order of rowmotion on a chain of claws

Let $m = \min(k,n)$. The order of rowmotion on $\mathcal{J}(\mathsf{C}_n \times [k])$ divides $m!(k+2)$.

The order in this Theorem is exact for most (but not all) pairs of n and k.

Theorem [PR24+]: Symmetry of statistics for non-minimal labels

Let $\chi_{(i,j)}$ be the indicator function for $(i,j) \in \mathsf{C}_n \times [k]$. Then for the action of rowmotion on $\mathcal{J}(\mathsf{C}_n \times [k])$, the statistic $\chi_{(i,a)} - \chi_{(j,a)}$ is 0-mesic for all $i, j \in [n]$ and $a \in [k]$.

Theorem [PR24+]: Homomesy of generalized flux capacitor configuration for claw posets

Let $B_i = \chi_{(i-1,\widehat{0})} + \sum_{j=1}^{n} \chi_{(i,j)}$. Then for the action of rowmotion on $\mathcal{J}(\mathsf{C}_n \times [k])$, $B_i - B_j$ is $\frac{(j-i)(n+1)}{k+2}$-mesic for all $i, j \in [n]$.

Selected References (See abstract for more references and details)

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