Rowmotion on the chain of V's poset, whirling dynamics and the flux-capacitor homomesy

Matthew Plante
(Joint work with Tom Roby)

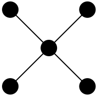
Slides available at `www.matthewplante.com`

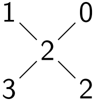
April 6th, 2025

Outline

Definition of whirling on posets

Let $\mathcal{F}_k$ be the set of *order-reversing function* from P to $\{0, 1, \ldots, k\}$. Also referred to as *P-partitions*.

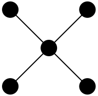
$P =$ 

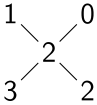


$\in \mathcal{F}_3(P)$

Definition of whirling on posets

Let $\mathcal{F}_k$ be the set of *order-reversing function* from P to $\{0, 1, \ldots, k\}$. Also referred to as P-*partitions*.

$P =$ 



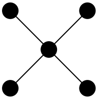
$\in \mathcal{F}_3(P)$

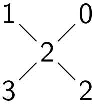
Definition

Let P be a poset. For $f \in \mathcal{F}_k(P)$ and $x \in P$ define $w_x : \mathcal{F}_k(P) \to \mathcal{F}_k(P)$, called the *whirl at* x, as follows: repeatedly add 1 $(\mathrm{mod}\ k+1)$ to the value of $f(x)$ until we get a function in $\mathcal{F}_k(P)$. This new function is $w_x(f)$.

Definition of whirling on posets

Let $\mathcal{F}_k$ be the set of *order-reversing function* from P to $\{0, 1, \ldots, k\}$. Also referred to as *P-partitions*.

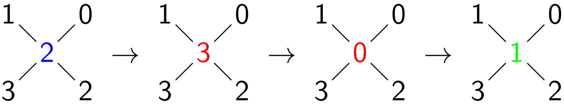
$P =$ 



$\in \mathcal{F}_3(P)$

Definition

Let P be a poset. For $f \in \mathcal{F}_k(P)$ and $x \in P$ define $w_x : \mathcal{F}_k(P) \to \mathcal{F}_k(P)$, called the *whirl at x*, as follows: repeatedly add 1 $\pmod{k+1}$ to the value of $f(x)$ until we get a function in $\mathcal{F}_k(P)$. This new function is $w_x(f)$.



Whirling

Proposition

If $x, y \in P$ are incomparable, then $w_x w_y(f) = w_y w_x(f)$.

Whirling

Proposition

If $x, y \in P$ are incomparable, then $w_x w_y(f) = w_y w_x(f)$.

Since x and y are incomparable, there are no inequalities constraining the relationship between $f(x)$ and $f(y)$. So $w_x w_y = w_y w_x$.

Whirling

Proposition

If $x, y \in P$ are incomparable, then $w_x w_y(f) = w_y w_x(f)$.

Since x and y are incomparable, there are no inequalities constraining the relationship between $f(x)$ and $f(y)$. So $w_x w_y = w_y w_x$.

Definition

Let $(x_1, x_2, \ldots, x_p)$ be a linear extension of P. Define $w : \mathcal{F}_k(P) \to \mathcal{F}_k(P)$ by $w := w_{x_1} w_{x_2} \ldots w_{x_p}$.

Whirling

Proposition

If $x, y \in P$ are incomparable, then $w_x w_y(f) = w_y w_x(f)$.

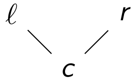
Since x and y are incomparable, there are no inequalities constraining the relationship between $f(x)$ and $f(y)$. So $w_x w_y = w_y w_x$.

Definition

Let $(x_1, x_2, \ldots, x_p)$ be a linear extension of P. Define $w : \mathcal{F}_k(P) \to \mathcal{F}_k(P)$ by $w := w_{x_1} w_{x_2} \ldots w_{x_p}$.

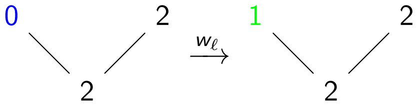
The above proposition shows that this is well-defined, since one can get from any linear extension to any other by a sequence of interchanges of incomparable elements.

Example of whirling V

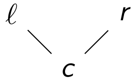
We whirl the example 

first at ℓ, r, then c.

Start with $(0, 2, 2) \in \mathcal{F}_2(\mathsf{V})$.

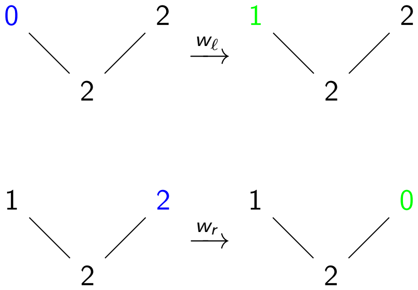


Example of whirling V

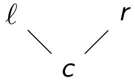
We whirl the example 

first at ℓ, r, then c.

Start with $(0, 2, 2) \in \mathcal{F}_2(\mathrm{V})$.

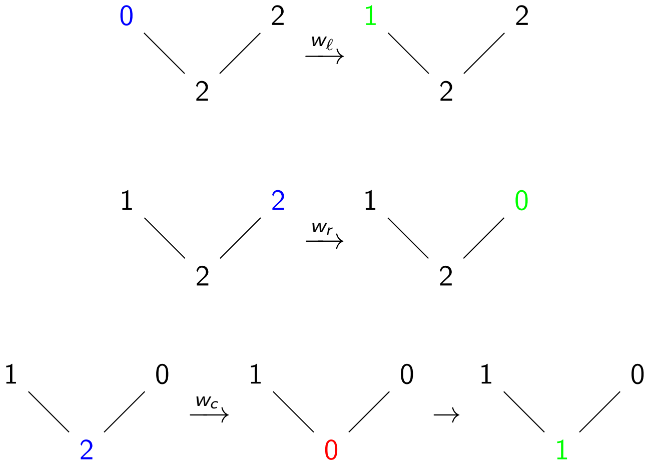


Example of whirling V

We whirl the example 

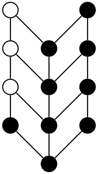
first at ℓ, r, then c.

Start with $(0,2,2) \in \mathcal{F}_2(\mathsf{V})$.

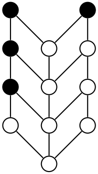


Order ideals

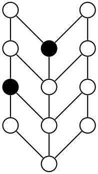
- Let $\mathcal{J}(P)$ denote the set of order ideals of a poset P.
  That is, for $I \subseteq V(n)$, $I \in \mathcal{J}_n$ if and only if $x \in I \implies y \in I$ for all $y \leq x$.



Order Ideal



Order Filter



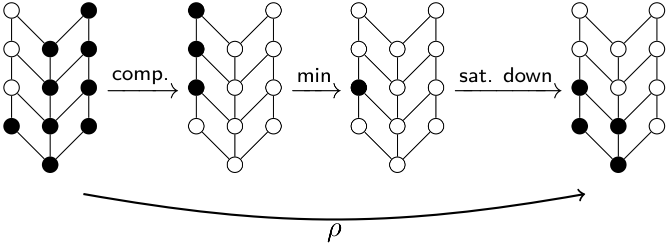
Antichain

Rowmotion on order ideals

- Denote *rowmotion* on order ideals by ρ. We define ρ on order ideals by taking the minimal elements of the complement and saturating down.

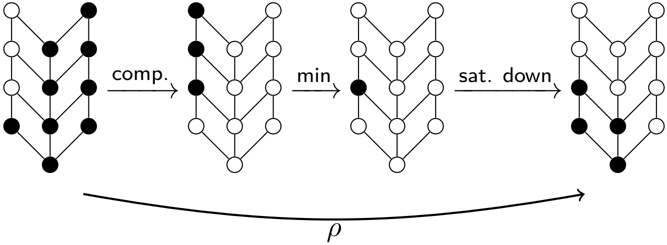
Rowmotion on order ideals

- Denote *rowmotion* on order ideals by ρ. We define ρ on order ideals by taking the minimal elements of the complement and saturating down.



Rowmotion on order ideals

- Denote *rowmotion* on order ideals by ρ. We define ρ on order ideals by taking the minimal elements of the complement and saturating down.



This map and its inverse have been considered with varying degrees of generality by many people: Duchet, Brouwer and Schrijver, Vameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams, who named it rowmotion.

Rowmotion as a product of toggles

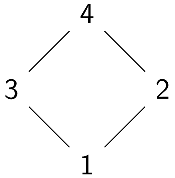
- Rowmotion has an alternate definition as a composition of toggling involutions by Cameron and Fon-der-Flaass [CaFl95]

$$\tau_i(I) = \begin{cases} I \setminus \{i\} & \text{if } i \in I \text{ and } I \setminus \{i\} \in \mathcal{J}(P) \\ I \cup \{i\} & \text{if } i \notin I \text{ and } I \cup \{i\} \in \mathcal{J}(P) \\ I & \text{otherwise.} \end{cases}$$

Rowmotion as a product of toggles

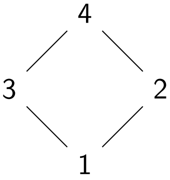
- Rowmotion has an alternate definition as a composition of toggling involutions by Cameron and Fon-der-Flaass [CaFl95]

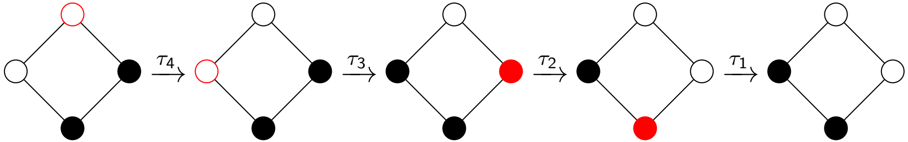
$$\tau_i(I) = \begin{cases} I \setminus \{i\} & \text{if } i \in I \text{ and } I \setminus \{i\} \in \mathcal{J}(P) \\ I \cup \{i\} & \text{if } i \notin I \text{ and } I \cup \{i\} \in \mathcal{J}(P) \\ I & \text{otherwise.} \end{cases}$$



- Rowmotion has an alternate definition as a composition of toggling involutions by Cameron and Fon-der-Flaass [CaFl95]

$$\tau_i(I) = \begin{cases} I \setminus \{i\} & \text{if } i \in I \text{ and } I \setminus \{i\} \in \mathcal{J}(P) \\ I \cup \{i\} & \text{if } i \notin I \text{ and } I \cup \{i\} \in \mathcal{J}(P) \\ I & \text{otherwise.} \end{cases}$$

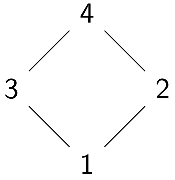


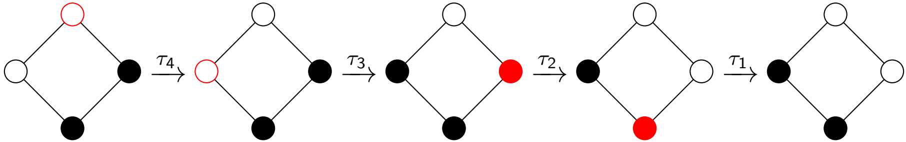


Rowmotion as a product of toggles

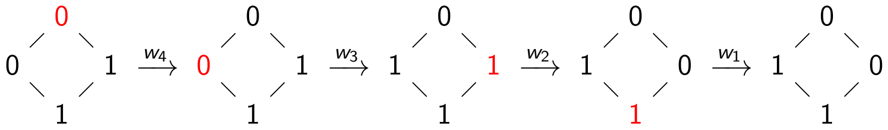
- Rowmotion has an alternate definition as a composition of toggling involutions by Cameron and Fon-der-Flaass [CaFl95]

$$\tau_i(I) = \begin{cases} I \setminus \{i\} & \text{if } i \in I \text{ and } I \setminus \{i\} \in \mathcal{J}(P) \\ I \cup \{i\} & \text{if } i \notin I \text{ and } I \cup \{i\} \in \mathcal{J}(P) \\ I & \text{otherwise.} \end{cases}$$





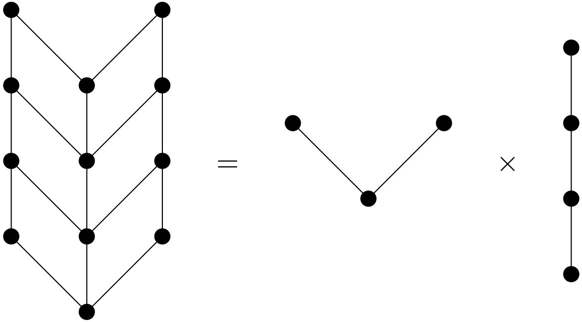
This coincides with whirling labeling from $\mathcal{F}_1(P)$.



Chain-factor posets

Definition

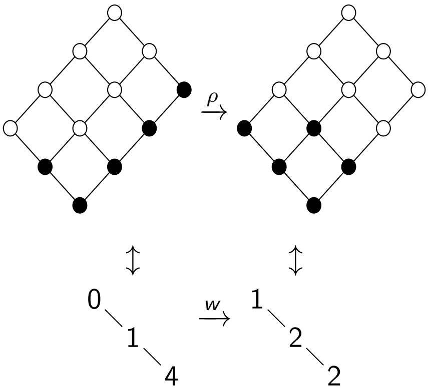
A *chain-factor poset* is a poset P such that $P = Q \times [n]$ for some poset Q.



Equivariant bijection between whirling and rowmotion

Theorem

There is an equivariant bijection between $\mathcal{F}_k(P)$ and $\mathcal{J}(P \times [k])$ (order ideals of chain-factor posets) which sends w to ρ.



Whorms

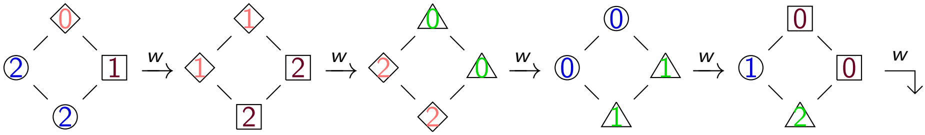
Definition

For any $x \in P$ and $f \in \mathcal{F}_k(P)$, define (x, f) to be a *whirl element*. The whirl element (y, g) is *whirl successive* to (x, f) if either:

1. $y = x$ and $g(y) = w(f)(x) = f(x) + 1$, or
2. x covers y, $f = g$, and $f(x) = g(y)$.

Two elements in a sequence of whirl successive elements are called *whorm-connected*. A *whorm* is a maximal set of whirl successive elements.

An orbit of whirling $\mathcal{F}_2(P)$ (for $P = [2] \times [2]$) with its four whorms indicated by the same color and (redundantly) node-shape.



Defining homomesy

Definition (Propp–Roby [PR15, Def. 1.1])

Let the invertible action τ act on the set S. Let f be the statistic $f : S \to K$. Assume every τ-orbit is finite. We say f is *homomesic* if there exists $c \in K$ such that

$$\frac{\sum_{s \in O} f(s)}{\# O} = c$$

for all orbits O. In such a case we say the triple $(\mathcal{S}, \tau, f)$ exhibit *homomesy* with average c.

Defining homomesy

Definition (Propp–Roby [PR15, Def. 1.1])

Let the invertible action τ act on the set S. Let f be the statistic $f : S \to K$. Assume every τ-orbit is finite. We say f is *homomesic* if there exists $c \in K$ such that

$$\frac{\sum_{s \in O} f(s)}{\# O} = c$$

for all orbits O. In such a case we say the triple $(\mathcal{S}, \tau, f)$ exhibit *homomesy* with average c.

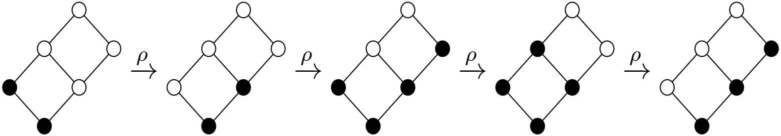
- If $\mathcal{S}$ is finite, then we can switch out c from the equation above with the global average.

$$\frac{\sum_{s \in O} f(s)}{\# O} = \frac{\sum_{s \in S} f(s)}{\# S}.$$

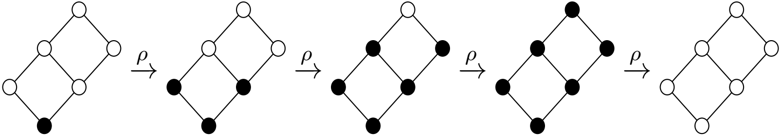
- If $(\mathcal{S}, \tau, f)$ exhibit *homomesy* with average c we say f is *c-mesic*.

Homomesy of product of two chains

Let χ_p be the indicator function of p in order ideal I. Consider the statistic $\sum_{p \in P} \chi_p$.



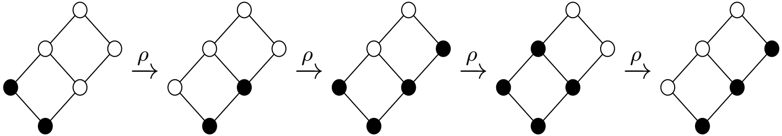
$$(2+2+4+4+3)/5 = 3$$



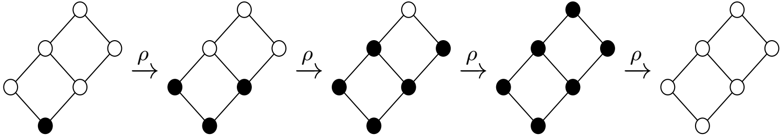
$$(1+3+5+6+0)/5 = 3$$

Homomesy of product of two chains

Let χ_p be the indicator function of p in order ideal I. Consider the statistic $\sum_{p \in P} \chi_p$.



$$(2 + 2 + 4 + 4 + 3)/5 = 3$$



$$(1 + 3 + 5 + 6 + 0)/5 = 3$$

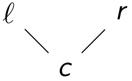
Theorem (Propp–Roby [PR15])

The action of rowmotion on $\mathcal{J}([a] \times [b])$ with cardinality statistic is $ab/2$-mesic.

Rowmotion on the chain of V's poset

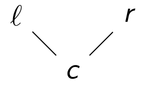
The poset $V \times [k]$

- Let V be the three-element partially ordered set with Hasse diagram

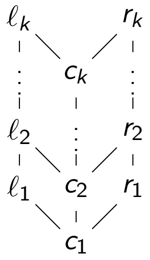


The poset V × [k]

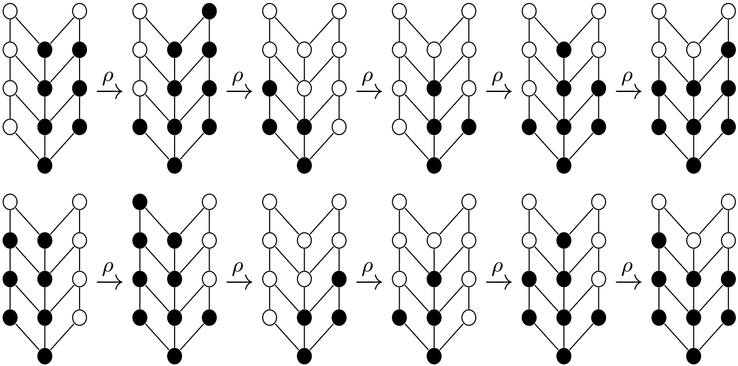
- Let V be the three-element partially ordered set with Hasse diagram



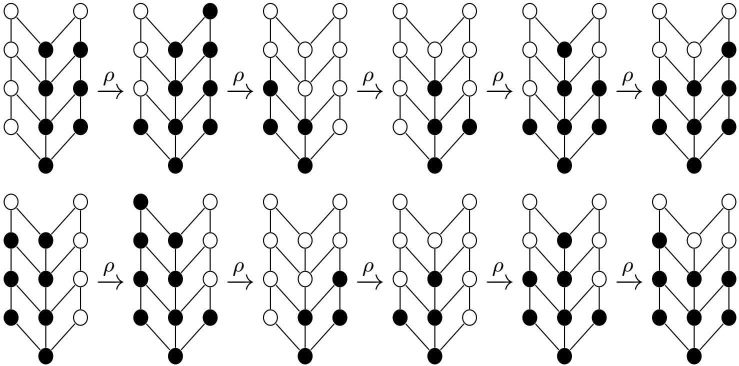
- The poset of interest is $V \times [k]$



Order-ideal rowmotion on V × $[k]$



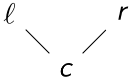
Order-ideal rowmotion on V × $[k]$



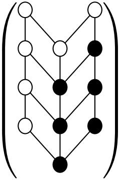
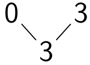
Theorem

Order ideals of V × [k] *are reflected about the center chain after* $k+2$ *iterations of* ρ*, and furthermore, the order of* ρ *on order ideals of* V × [k] *is* $2(k+2)$.

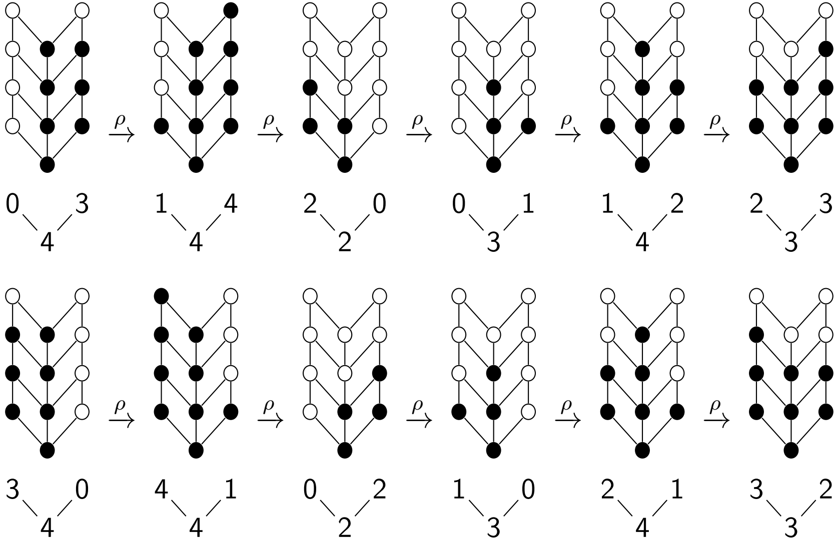
Map to order-reversing functions on V



1. Define $\mathcal{F}_k(\mathsf{V}) = \{(\ell, c, r) \in \{0, \ldots, k\}^3 : \ell, r \leq c\}$.
2. Define $\phi : \mathcal{J}(\mathsf{V} \times [k]) \to \mathcal{F}_k(\mathsf{V})$ by $\phi(I) = \left(\sum \chi_{\ell_i}, \sum \chi_{c_i}, \sum \chi_{r_i}\right)$.

ϕ  $= (0, 3, 3) \leftrightarrow$ 

Example of rowmotion orbit with triples



Periodicity

Theorem

Order ideals of $\mathsf{V} \times [\mathsf{k}]$ *are reflected about the center chain after* $k+2$ *iterations of* ρ*, and furthermore, the order of* ρ *on order ideals of* $\mathsf{V} \times [\mathsf{k}]$ *is* $2(k+2)$.

Direct inspection of order-reversing functions on V gives a straightforward proof of periodicity. However a study of *whorms* gives a deeper understanding of the orbit structure.

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

0	2	0
1	3	1
2	4	2
3	3	3
0	4	0
1	1	1

Tiling an orbit board with whorms

Definition

For any $x \in P$ and $f \in \mathcal{F}_k(P)$, define (x, f) to be a *whirl element*. The whirl element (y, g) is *whirl successive* to (x, f) if either:

1. $y = x$ and $g(y) = w(f)(x) = f(x) + 1$, or
2. x covers y, $f = g$, and $f(x) = g(y)$.

Two elements in a sequence of whirl successive elements are called *whorm-connected*. A *whorm* is a maximal set of whirl successive elements.

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

0	2	0
1	3	1
2	4	2
3	3	3
0	4	0
1	1	1

The red whorm is a *left whorm* and a *one-tailed whorm*, the green whorm is a *right whorm* and one-tailed whorm, and the blue whorm is a *two-tailed whorm*. For a whorm ξ, let $\mathrm{h}(\xi)$ be the number of rows intersecting the center column, and let $\mathrm{t}(\xi)$ be the number of rows intersecting the outer columns.

Center-seeking whorms

Theorem

Any orbit board of $\mathcal{F}_k(V)$ can be decomposed into 6 one-tailed whorms of length $k+2$ (or 2 two-tailed whorms if the functions are symmetric.) We call these whorms, center-seeking whorms.

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

0	2	0
1	3	1
2	4	2
3	3	3
0	4	0
1	1	1

Center-seeking whorms

Theorem

Any orbit board of $\mathcal{F}_k(\mathrm{V})$ can be decomposed into 6 one-tailed whorms of length $k+2$ (or 2 two-tailed whorms if the functions are symmetric.) We call these whorms, center-seeking whorms.

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

0	2	0
1	3	1
2	4	2
3	3	3
0	4	0
1	1	1

Using the lemma

Given a whorm ξ in an orbit board of $\mathcal{F}_k(\mathsf{V})$, it can reasoned $\mathsf{h}(\xi) + \mathsf{t}(\xi) = \mathsf{k} + 2$. We will place a circular order on the whorms. Let ξ_1 and ξ_2 be whorms in an orbit board of $\mathcal{F}_k(\mathsf{V})$. If there exists $(c, f) \in \xi_1$ with $f(c) = k$ such that $(c, w(f)) \in \xi_2$, then we say ξ_2 is *in front of* ξ_1. We call a sequence of whorms *consecutive* if each is in front of the next.

Lemma

Given an orbit board $\mathcal{R}$ of w on $\mathcal{F}_k(\mathsf{V})$, let ξ_1, ξ_2, and ξ_3 be three consecutive whorms (not necessarily all distinct), that is, ξ_3 is in front of ξ_2 which is in front of ξ_1 in $\mathcal{R}$.

1. *If $\mathcal{R}$ is tiled entirely by one-tailed whorms, then*
$$\mathsf{t}(\xi_1) + \mathsf{t}(\xi_2) + \mathsf{t}(\xi_3) = 2(\mathsf{k} + 2).$$
2. *Otherwise, if $\mathcal{R}$ is tiled entirely by two-tailed whorms, then*
$$\mathsf{t}(\xi_1) + \mathsf{t}(\xi_2) = \mathsf{k} + 2.$$

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

Using the lemma 2

Given a whorm ξ in an orbit board of $\mathcal{F}_k(\mathrm{V})$, it can reasoned $\mathrm{h}(\xi) + \mathrm{t}(\xi) = \mathrm{k} + 2$. We will place a circular order on the whorms. Let ξ_1 and ξ_2 be whorms in an orbit board of $\mathcal{F}_k(\mathrm{V})$. If there exists $(c, f) \in \xi_1$ with $f(c) = k$ such that $(c, w(f)) \in \xi_2$, then we say ξ_2 is *in front of* ξ_1. We call a sequence of whorms *consecutive* if each is in front of the next.

Lemma

Given an orbit board with one-tailed whorms, let ξ_1, ξ_2, ξ_3, and ξ_4 be consecutive, then

$$\mathrm{t}(\xi_4) = \mathrm{t}(\xi_1).$$

Otherwise, if the orbit board contains two-tailed whorms, then $\mathrm{t}(\xi_1) = \mathrm{t}(\xi_3)$.

Theorem

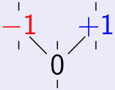
Let $(x, y, z) \in \mathcal{F}_k(\mathrm{V})$, then $w^{k+2}(x, y, z) = (z, y, x)$.

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

Homomesy for V

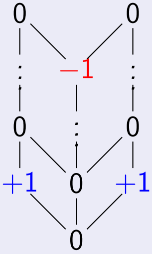
Theorem

For the action of rowmotion on order ideals of $\mathsf{V} \times [k]$*:*

1. *The statistic* $\chi_{r_i} - \chi_{\ell_i}$ *is 0-mesic* 

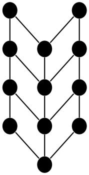
for each $i = 1, \ldots, k$*,*

   where χ_p *is the indicator function.*
2. *The statistic* $\chi_{\ell_1} + \chi_{r_1} - \chi_{c_k}$ *is* $\frac{2(k-1)}{k+2}$*-mesic.*



.

Sketch of proof of homomesy



$$\sum \chi_{\ell_1} + \chi_{r_1} - \chi_{c_k}$$

$$= (2(k+2)-3)+(2(k+2)-3)-6$$

Thus we see

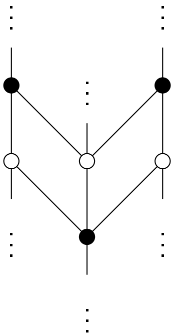
$$\frac{4(k+2)-12}{2(k+2)} = \frac{2k-2}{k+2}.$$

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

$2(k+2)$ rows

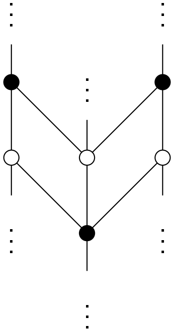
Flux capacitor

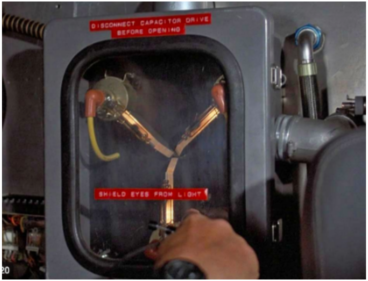
Let $F_i = \chi_{\ell_i} + \chi_{r_i} + \chi_{c_{i-1}}$, which has the following *flux-capacitor* shape in $V \times [k]$.



Flux capacitor

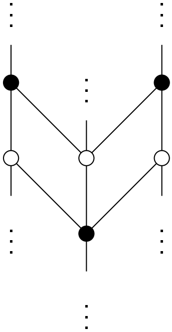
Let $F_i = \chi_{\ell_i} + \chi_{r_i} + \chi_{c_{i-1}}$, which has the following *flux-capacitor* shape in $V \times [k]$.

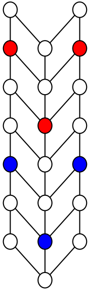




The flux capacitor homomesy

Let $F_i = \chi_{\ell_i} + \chi_{r_i} + \chi_{c_{i-1}}$, which has the following *flux-capacitor* shape in $V \times [k]$.





$F_3 - F_6$

Theorem

*For $k > 1$. The difference of **symmetrically-placed** flux-capacitor statistics, $F_i - F_{k+2-i}$, is $\frac{3(k+2-2i)}{k+2}$-mesic.*

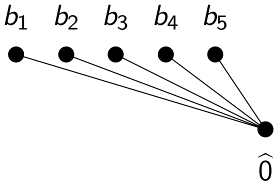
Rowmotion on the chain of claw's poset

Claw poset

Definition

We define the *claw poset* $C_n = \{b_1, \ldots, b_n, \widehat{0}\}$ where each b_i covers $\widehat{0}$. The *chain of claws poset* is defined to be $C_n \times [k]$.

For example, the Hasse diagram of C_5 would be



Orbit of whirling $\mathcal{F}_3(C_5)$ with whorms high-lighted.

b_1	b_2	b_3	b_4	b_5	$\widehat{0}$
1	1	0	1	2	3
2	2	1	2	3	3
3	3	2	3	0	3
0	0	3	0	1	3
1	1	0	1	2	2
2	2	1	2	0	3
3	3	2	3	1	3
0	0	3	0	2	3
1	1	0	1	3	3
2	2	1	2	0	2
0	0	2	0	1	3
1	1	3	1	2	3
2	2	0	2	3	3
3	3	1	3	0	3
0	0	2	0	1	2

Label restricted whirling

Definition

For any $A \subseteq [0, k]$, define the family of order-reversing maps

$$\mathcal{F}_k^A(\mathsf{C}_n) = \{f : f \in \mathcal{F}_k(\mathsf{C}_n) \text{ and } f(b_j) \in A \text{ for all } j \in [n]\}.$$

Then set $\overline{w}_A := \mathcal{F}_k^A(\mathsf{C}_n)$ to be the map which whirls (cyclically increments *within the subset* A) each label on the non-$\widehat{0}$ elements of C_n.

Consider $f = (1, 3, 3, 0, 4, 1, 6) \in \mathcal{F}_9(\mathsf{C}_6)$. We see $A(f) = \{0, 1, 3, 4\}$ so

$$\overline{w}_{A(f)}(1, 3, 3, 0, 4, 1, 6) = (3, 4, 4, 1, 0, 3, 6).$$

The last entry remains unchanged, and the earlier entries are increasing cyclically within the set $A(f) = \{0, 1, 3, 4\}$, with $\alpha := \#A = 4$.

Order of rowmotion on chain of claws

Theorem

Let w be the whirling action on k-bounded P-partitions in $\mathcal{F}_k(\mathsf{C}_n)$. For any $f \in \mathcal{F}_k(\mathsf{C}_n)$ with $A = A(f)$ and $\alpha = \alpha(f) = \#A(f)$, we have $w^{k+2} f = \overline{w}_{A(f)} f$. Thus, $w^{\alpha(k+2)} f = f$.

Similar to the $\mathsf{V} \times [k]$ case, this theorem is proved using whorms.

Theorem

Let $m = \min(k+1, n)$. The order of rowmotion on the chain of claws poset $\mathcal{J}(\mathsf{C}_n \times [k])$ is $(k+2)\operatorname{lcm}(1, 2, \ldots, m)$.

b_1	b_2	b_3	b_4	b_5	$\widehat{0}$
1	1	0	1	2	3
2	2	1	2	3	3
3	3	2	3	0	3
0	0	3	0	1	3
1	1	0	1	2	2
2	2	1	2	0	3
3	3	2	3	1	3
0	0	3	0	2	3
1	1	0	1	3	3
2	2	1	2	0	2
0	0	2	0	1	3
1	1	3	1	2	3
2	2	0	2	3	3
3	3	1	3	0	3
0	0	2	0	1	2

Homomesies for claw poset

Theorem

Let $\chi_{(i,a)}$ denote the indicator function for $(b_i, a) \in \mathsf{C}_n \times [k]$. Then for the action of rowmotion on $\mathcal{J}(\mathsf{C}_n \times [k])$, the statistic $\chi_{(i,a)} - \chi_{(j,a)}$ is 0-mesic for all $i, j \in [n]$ and $a \in [k]$.

The "flux-capacitor" homomesy fails to generalize to the claw-graph setting.

Future directions

Future projects include whirling investigations of

1. chains of minuscule posets,
2. chains of fence posets,
3. chains of zig-zag posets, and
4. product of three-chains.

Thank You!

[CaFl95] Peter J. Cameron and Dmitry G. Fon-der-Flaass, *Orbits of Antichains Revisited*, Europ. J. of Combin., **16(6)**, (1995), 545–554, http://www.sciencedirect.com/science/article/pii/0195669895900365.

[SW11] Jessica Striker and Nathan Williams, *Promotion and Rowmotion*, Europ. J. of Combin. 33 (2012), 1919–1942,

[PR15] James Propp and Tom Roby, *Homomesy in products of two chains*, Electronic J. Combin. **22(3)** (2015), #P3.4, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p4.

[R16] Tom Roby, *Dynamical algebraic combinatorics and the homomesy phenomenon* in A. Beveridge, et. al., Recent Trends in Combinatorics, IMA Volumes in Math. and its Appl., **159** (2016), 619–652.

[JR18] Michael Joseph and Tom Roby, *Toggling Independent Sets of a Path Graph*, Electronic Journal of Combin., **25(1)**, 2018, #P1.18.

[JPR18] Michael Joseph and James Propp and Tom Roby, *Whirling injections, surjections, and other functions between finite sets*, 2018. https://arxiv.org/abs/1711.02411

[EPRS] Sergi Elizalde and Matthew Plante and Tom Roby and Bruce Sagan, *Rowmotion on fences*, arXiv:2108.12443.