Toggling Independent Sets of a Complete Graph Cross a Chain

Matthew Plante

Slides available at `www.matthewplante.com`

June 23 2020

Outline

1. Binary Strings and an Example of Homomesy
2. Toggling Independent Sets on a Graph
3. " " on a Path Graph
4. " " on a Rectangle Graph
5. " " on $K_m \times [n]$
6. Relating these Maps to Promotion of Order Ideals in Posets

Binary Strings and an Example of Homomesy

Binary Strings

- Let $S_{n,k}$ be the set of strings of 0s and 1s with length n and exactly k 1s.
- Let τ be rightward cyclic shift that bumps each entry to the right and the last entry to the front

Example ($S_{7,3}$)

$$\tau(0110101) = 1011010$$

- τ is periodic (order divides n)
- τ is invertible (τ^{-1} =leftward cyclic shift)

τ-Orbits

- Since τ is periodic and invertible every element of $S_{n,k}$ belongs to a unique τ-orbit.
- We can naturally partition $S_{n,k}$ into τ-**orbit boards**. Where below each string is τ applied to that string.

Example (The four τ-orbit boards of $S_{6,3}$)

000111	001101	001011	010101
100011	100110	100101	101010
110001	010011	110010	
111000	101001	011001	
011100	110100	101100	
001110	011010	010110	

- These boards are infinite but we will write just one period. We sometimes call a board with more than one period a **superorbit board**.
- When there is no ambiguity we just call them orbit boards.

Inversions

- An **inversion** in a binary string s is a pair (i, j) with $i < j$ and $s_i > s_j$. In this case, $s_i = 1$ but $s_j = 0$.

- Let $f : S_{n,k} \to \mathbb{Q}$ be the number of inversions in any string in $S_{n,k}$

Example ($S_{7,3}$)

$$f(0110010) = 0 + 2 + 2 + 3 = 7$$

Same Average

- If we count calculate f for each string in $S_{n,k}$ we notice a surprising phenomenon.

Example (The four τ-orbit boards of $S_{6,3}$ with number of inversions)

000111 0	001101 2	001011 1	010101 3
100011 3	100110 5	100101 4	101010 6
110001 6	010011 2	110010 7	
111000 9	101001 5	011001 4	
011100 6	110100 8	101100 7	
001110 3	011010 5	010110 4	

- If we calculate the average number of inversions per orbit board we get: $\frac{0+3+6+9+6+3}{6} = \frac{2+5+2+5+8+5}{6} = \frac{1+4+7+4+7+4}{6} = \frac{3+6}{2} = 4.5$ Same average!
- The set $S_{n,k}$ under the τ-orbit with statistic f is a **homomesy**, and we say f is 4.5-mesic.

Defining Homomesy

Let

- $\mathcal{S}$ be a set,
- $\tau : \mathcal{S} \to \mathcal{S}$ be an invertible map,
- K be a field of characteristic 0.

Definition

A *τ-orbit* is an equivalence class on $\mathcal{S}$ under the relation $\sim$, where $s \sim t$ provided $s = \tau^j t$ for some $j \in \mathbb{N}$.

Definition

A *statistic* is any map $f : \mathcal{S} \to K$.

Defining Homomesy Cont.

Definition ([PR15, Def. 1.1])

Assume every τ-orbit is finite. We say f is *homomesic* if there exists $c \in K$ such that

$$\frac{\sum_{s \in O} f(s)}{\# O} = c$$

for all orbits O. In such a case we say the triple $(\mathcal{S}, \tau, f)$ exhibit *homomesy* with average c.

- If $\mathcal{S}$ is finite then we can switch out c from the equation above with the global average.

$$\frac{\sum_{s \in O} f(s)}{\# O} = \frac{\sum_{s \in S} f(s)}{\# S}.$$

Example: Binary Strings Cont.

Let

- $\mathcal{S} = S_{n,k}$
- τ is rightward cyclic shift
- $f(s) = \#$ of inversions.

Theorem ([PR15, Thm. 2.3])

The triple $(S_{n,k}, \tau, f)$ exhibit homomesy with average $k(n-k)/2$.

- Our first example was the inversion statistic on rightward cyclic rotation of binary strings for $n = 6$ and $k = 3$.

Example: Binary Strings Cont.

Another statistic, take χ_j, to be the jth 1 indicator function so

$$\chi_3(111000) = 1, \qquad \chi_3(100110) = 0$$

Example (The four τ-orbit boards of $S_{6,3}$ with χ_3)

000111 0	001101 1	001011 1	010101 0
100011 0	100110 0	100101 0	101010 1
110001 0	010011 0	110010 0	
111000 1	101001 1	011001 1	
011100 1	110100 0	101100 1	
001110 1	011010 1	010110 0	

- So $(S_{6,3}, \tau, \chi_3)$ is .5-mesic.
- In general $(S_{n,k}, \tau, \chi_j)$ is k/n-mesic.

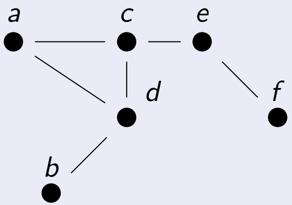
Toggling Independent Sets on a Graph

Graph Notation

Let $G = (V, E)$ be an undirected graph with no loops.

- V is the vertex set.
- E is the edge set, a set of pairs $\{u, v\}$ where $u, v \in V$. For simplicity we'll write $uv \in E$ when convenient instead of $\{u, v\} \in E$.

Example (A Graph with 6 vertices)



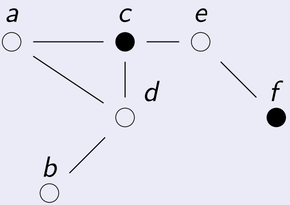
- $V = \{a, b, c, d, e, f\}$
- $E = \{ac, ad, bd, cd, ce, ef\}$

- If $\{u, v\} \in E$ then u and v are **adjacent**. Specifically we can say u is adjacent to v and v is adjacent to u. Also say u and v are neighbors.

Independent Sets of a Graph

We say a set $S \subset V$ is **independent** if $uv \notin E$ for all distinct pairs $u, v \in S$.

Example

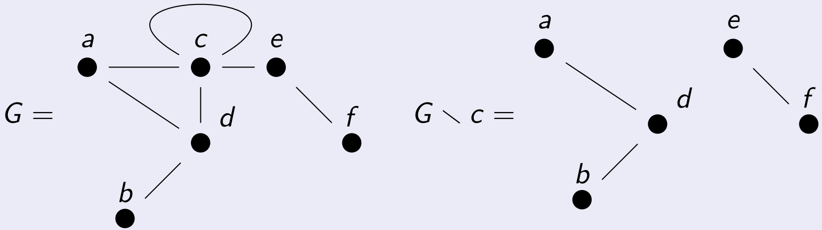


- *We shade vertices* $\bullet$ *to indicates the vertex is in* S *and* $\circ$ *to indicates the vertex is not in* S.
- *This set,* $S = \{c, f\}$*, is independent.*

Why No Loops

- If a graph G has a loop at vertex v then v can never be in an independent set since v is adjacent to itself.
- As sets, $\mathcal{I}(G)$ would be the same as $\mathcal{I}(G \setminus v)$ so we will assume our graphs do not have loops.

Example



$$\mathcal{I}(G) = \mathcal{I}(G \setminus c) = \{\{a\}, \{b\}, \{d\}, \{e\}, \{f\}, \{a,b\}, \{a,e\}, \{a,f\}, \{a,b,e\}, \{a,b,f\}, \{b,e\}, \{b,f\}, \{d,e\}, \{d,f\}\}$$

Toggling on Independent Sets

- Let $\mathcal{I}(G)$ denote the set of all independent set of the graph G.
- Let $G = (V, E)$ and $S \in \mathcal{I}(G)$.
- For all $v \in V$, let $\tau_v(S) : \mathcal{I}(G) \to \mathcal{I}(G)$ denote the following map

$$\tau_v(S) = \begin{cases} S \setminus (v) & \text{if } v \in S, \\ S \cup v & \text{if } v \notin S \text{ and } S \cup v \in \mathcal{I}(G), \\ S & \text{otherwise.} \end{cases}$$

  we call τ_v the **toggle** at v.
- Notice that $\tau_v^2(S) = S$ for all $S \in \mathcal{I}(G)$ and therefore τ_v is an involution.

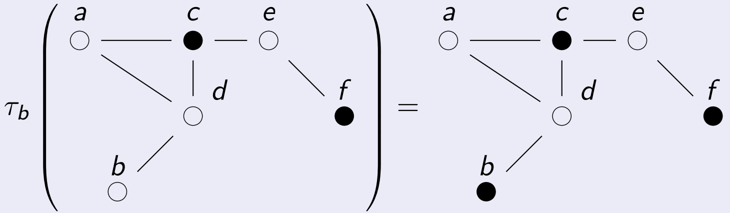
Theorem (Generalization of [JR18, Thm. 2.2])

For graph $G = (V, E)$, two toggles τ_v and τ_u commute if and only if $uv \notin E$.

Toggling cont.

$$\tau_v(S) = \begin{cases} S \setminus (v) & \text{if } v \in S, \\ S \cup v & \text{if } v \notin S \text{ and } S \cup v \in \mathcal{I}(G), \\ S & \text{otherwise.} \end{cases}$$

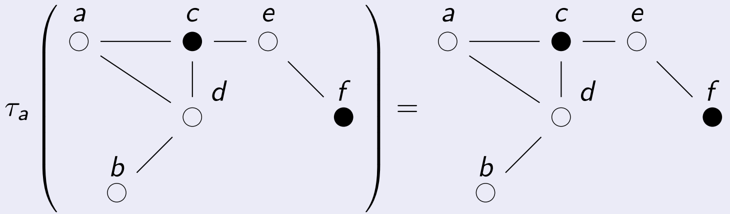
Example ($\tau_b(\{c, f\})$)



Toggling cont.

$$\tau_v(S) = \begin{cases} S \setminus (v) & \text{if } v \in S, \\ S \cup v & \text{if } v \notin S \text{ and } S \cup v \in \mathcal{I}(G), \\ S & \text{otherwise.} \end{cases}$$

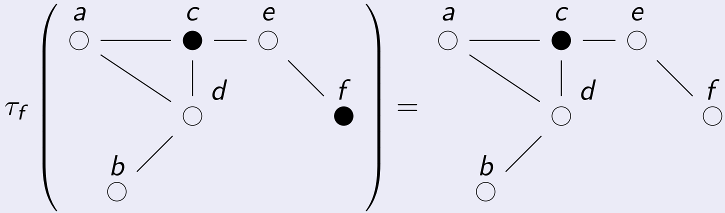
Example ($\tau_a(\{c, f\})$)



Toggling cont.

$$\tau_v(S) = \begin{cases} S \setminus (v) & \text{if } v \in S, \\ S \cup v & \text{if } v \notin S \text{ and } S \cup v \in \mathcal{I}(G), \\ S & \text{otherwise.} \end{cases}$$

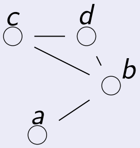
Example ($\tau_f(\{c, f\})$)



Leaves

- A **leaf** is a vertex which is only adjacent to exactly one vertex.

Example (A graph with one leaf.)



The vertex a is a leaf.

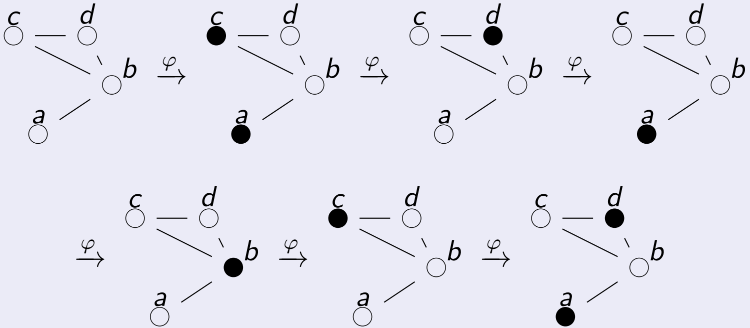
- Every leaf has exactly one neighbor so the result of toggling at a leaf depends entirely on whether or not either of these vertices are in the set.

General Toggling Result

Theorem (Generalization of [JR18, Thm. 2.12])

Let $G = (V, E)$ be a graph and $v \in V$ be a leaf with nonleaf neighbor $u \in V$. Let φ be a product of toggles such that τ_v and τ_u are in the product once each. The triple $(\mathcal{I}(G), \varphi, 2\chi_v + 1\chi_u)$ exhibit homomesy with average 1.

Example ($\varphi = \tau_d \tau_c \tau_b \tau_a$)



General Toggling Result

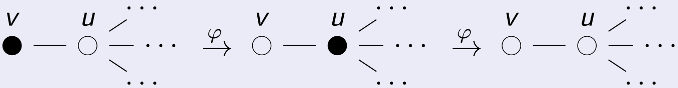
Theorem (Generalization of [JR18, Thm. 2.12])

Let $G = (V, E)$ be a graph and $v \in V$ be a leaf with nonleaf neighbor $u \in V$. Let φ be a product of toggles such that τ_v and τ_u are in the product once each. The triple $(\mathcal{I}(G), \varphi, 2\chi_v + 1\chi_u)$ exhibit homomesy with average 1.

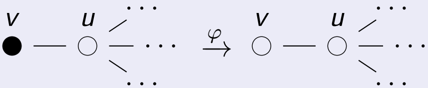
Sketch of Proof.

Assuming τ_v comes before τ_u in φ and S contains v but not u.

Case 1.



Case 2.



Similar case if τ_v comes before τ_u in φ.

□

Toggling Independent Sets on a Path Graph

Path Graphs

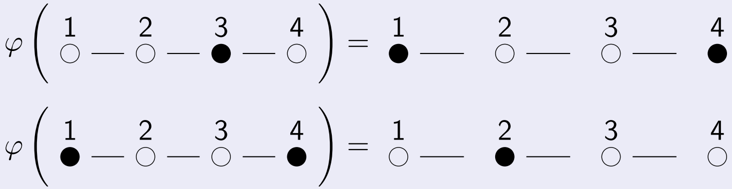
Let $P_n = (V, E)$ be the path graph with $V = [n] = \{1, \ldots, n\}$ and $E = \{\{i, i+1\} : i < n\}$.

- We can represent the independent set $\{3, 6\}$ with the binary string 001001 where a 1 in the jth spot means j is in the set.
  - Sometimes called the characteristic vector.
- The number of binary strings of length n that do not contain substrings of 11 is easily given by the nth Fibonacci number.

Left to Right Toggling

- Let φ be the toggle of each vertex of P_n from let to right.
- For P_4, $\varphi = \tau_4\tau_3\tau_2\tau_1$.

Example (Toggling an independent set from $\mathcal{I}(P_4)$)



- In binary notation we write $\varphi(0010) = 1001$ and $\varphi(1001) = 0100$.

φ-Orbit Board and Homomesy

Example (The φ-orbit boards of $\mathcal{I}(P_5)$)

00000	01000	10010
10101	00101	01001
	10000	00100
	01010	
	00001	
	10100	
	00010	
	10001	
	01000	

- Toggling independent sets of the path graph results in orbits which are much less regular than for cyclic rotation structure.

φ-Orbit Board and Homomesy

Example (The φ-orbit boards of $\mathcal{I}(P_5)$)

00000	01000	10010
10101	00101	01001
	10000	00100
	01010	
	00001	
	10100	
	00010	
	10001	
	01000	
10101	32223	11111

Theorem ([JR18, Thm. 2.9])

The triple $(\mathcal{I}(P_n), \varphi, \chi_j - \chi_{n+1-j})$ *exhibits homomesy with average 0 for all* $1 \leq j \leq n$.

Partition Orbit Board into Snakes

1	0	0	0	1	0	1	0	1	0
0	1	0	0	0	0	0	0	0	1
0	0	1	0	1	0	1	0	0	0
1	0	0	0	0	0	0	1	0	1
0	1	0	1	0	1	0	0	0	0
0	0	0	0	0	0	1	0	1	0
1	0	1	0	1	0	0	0	0	1
0	0	0	0	0	1	0	1	0	0
1	0	1	0	0	0	0	0	1	0
0	0	0	1	0	1	0	0	0	1
1	0	0	0	0	0	1	0	0	0
0	1	0	1	0	0	0	1	0	1
0	0	0	0	1	0	0	0	0	0
1	0	1	0	0	1	0	1	0	1
0	0	0	1	0	0	0	0	0	0
6	3	4	4	4	4	4	4	3	6

Sketch of Proof.

Partition the orbit board into snakes. Whenever we have a 1 in the oribit board that is not in the right column exactly one of two things happen.

- There is a 1 two places to the right.

  1 0 1

- Otherwise there is a 1 down and to the right

  1 0 0
  0 1 0

□

Cyclic Shifting Snakes

1	0	0	0	1	0	1	0	1	0
0	1	0	0	0	0	0	0	0	1
0	0	1	0	1	0	1	0	0	0
1	0	0	0	0	0	0	1	0	1
0	1	0	1	0	1	0	0	0	0
0	0	0	0	0	0	1	0	1	0
1	0	1	0	1	0	0	0	0	1
0	0	0	0	0	1	0	1	0	0
1	0	1	0	0	0	0	0	1	0
0	0	0	1	0	1	0	0	0	1
1	0	0	0	0	0	1	0	0	0
0	1	0	1	0	0	0	1	0	1
0	0	0	0	1	0	0	0	0	0
1	0	1	0	0	1	0	1	0	1
0	0	0	1	0	0	0	0	0	0

Sketch of Proof.

Let 2 denote a jump to the right and 1 diagonal down-right move. We get right-cylic shifts of compositions of 9.

112212

122121

221211

212112

121122

211221

So with one composition of 9 we can recover the entire board and know where the 1s will be. □

Other Properties of the Path Graph

We can also use this snake decomposition to get other properties of the orbit board.

- Can be used to determine the sizes of orbit boards.
- The number of orbits is the number of inequivalent compositions of $n-1$ into 1s and 2s. (Number of binary necklaces avoiding substring 00)
- With a fair amount of work and Coxeter theory the following can be proved

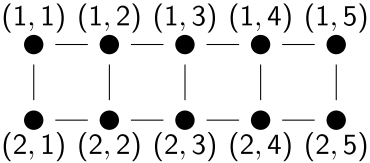
Theorem ([JR18, Thm. 2.30])

Let ψ be product of the toggles $\tau_1, \ldots, \tau_n$ in any order. Any statistic which is a linear combination of the indicator functions χ_j is c-mesic under the action of φ if and only if it is c-mesic under the action of ψ.

Toggling Independent Sets on a Rectangle Graph

Rectangle Graph

Let $R_n = (V, E)$ be the $2 \times n$ **rectangle graph** where $V = \{(i,j) | 1 \le i \le 2, 1 \le j \le n\}$ and $\{(i,j),(k,\ell)\} \in E$ if and only if $i = k$ and $|j - \ell| = 1$, or $i \ne k$ and $j = \ell$.

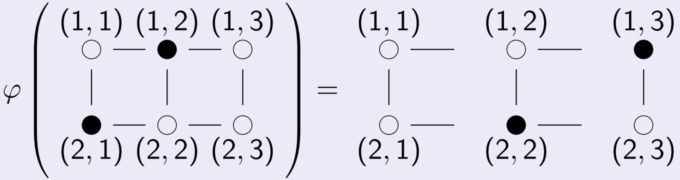


- Goal: Prove a homomesy for $\mathcal{I}(R_n)$ the independent sets of the rectangle $2 \times n$ similar to that for the $\mathcal{I}(P_n)$ independent sets of the $1 \times n$ rectangle.

Toggle on R_n

- Define $\varphi = \tau_{(2,n)}\tau_{(1,n)} \cdots \tau_{(2,1)}\tau_{(1,1)}$, toggling top to bottom, left to right.

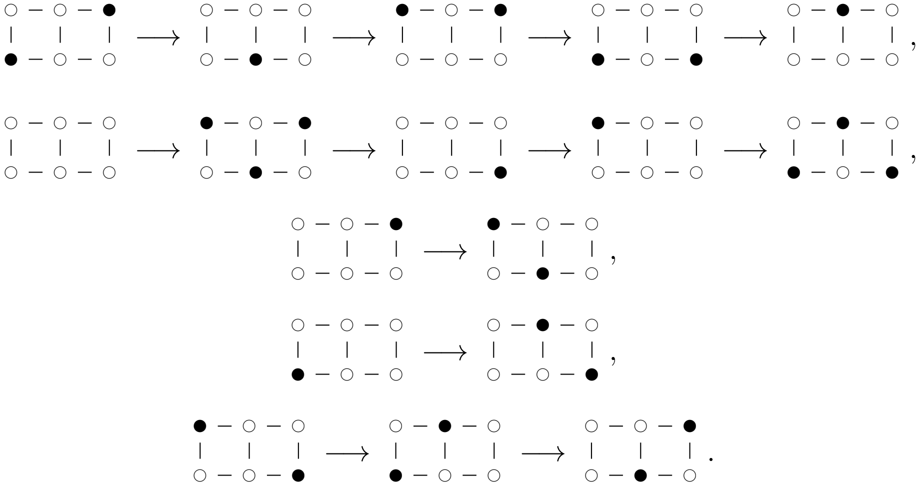
Example



- By commutativity of toggles $\varphi = \tau_{(2,n)} \cdots \tau_{(2,1)}\tau_{(1,n)} \cdots \tau_{(1,1)}$ toggling left to right then top to bottom.

φ-Orbits of $\mathcal{I}(R_3)$

The five orbits of $\mathcal{I}(R_3)$ under $\varphi = \tau_{(2,n)}\tau_{(1,n)} \cdots \tau_{(2,1)}\tau_{(1,1)}$



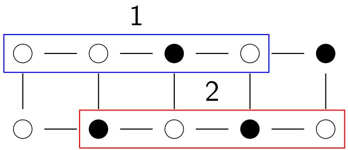
Homomesy 1

Conjecture (Homomesy 1)

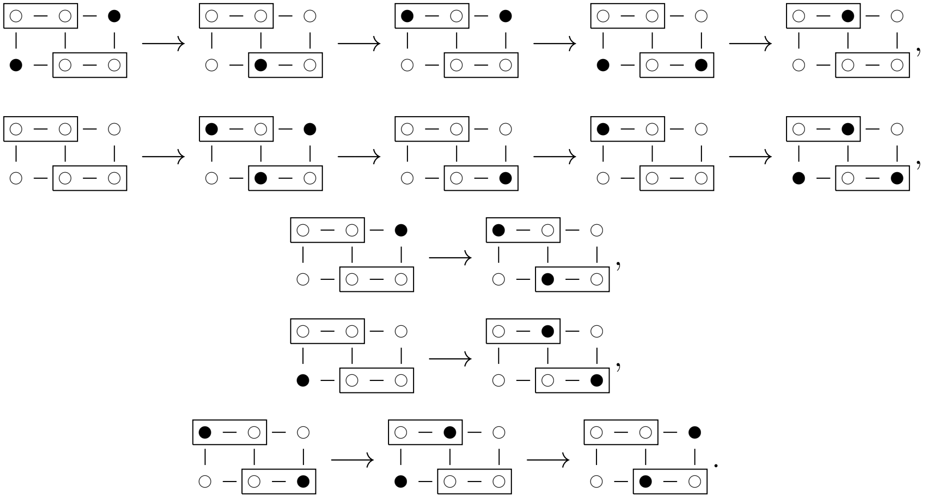
For $\mathcal{I}(R_n)$ with $n \geq 2$ given and

$$f(S) = \sum_{i=1}^{n-1} \chi_{(1,i)} - \sum_{i=2}^{n} \chi_{(2,i)}$$

the triple $(\mathcal{I}(R_n), \varphi, f)$ exhibit homomesy with average 0.



Example of Homomesy 1 in $\mathcal{I}(R_3)$



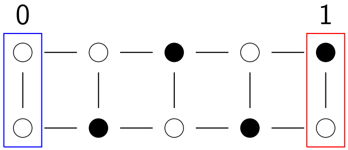
Homomesy 2

Conjecture (Homomesy 2)

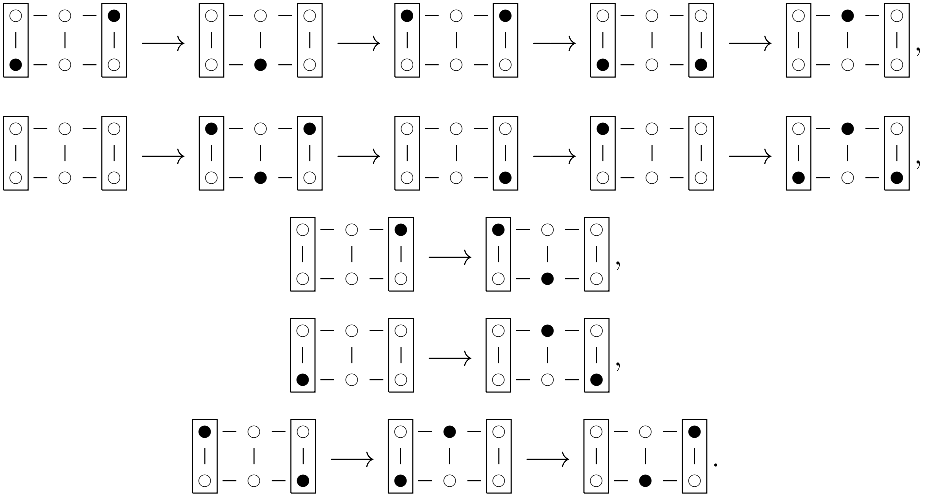
For $\mathcal{I}(R_n)$ with $n \geq 2$ given and let

$$f(S) = \chi_{(1,1)} + \chi_{(2,1)} - (\chi_{(1,n)} + \chi_{(2,n)})$$

then the triple $(\mathcal{I}(R_n), \varphi, f)$ exhibit homomesy with average 0.



Example of Homomesy 2 in $\mathcal{I}(R_3)$



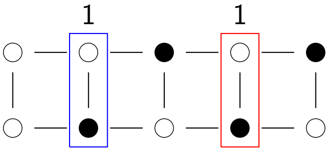
Homomesy 3

Conjecture (Homomesy 3)

For $\mathcal{I}(R_n)$ with $n \geq 2$ given and

$$f(S) = \chi_{(1,2)} + \chi_{(2,2)} - (\chi_{(1,n-1)} + \chi_{(2,n-1)})$$

then the triple $(\mathcal{I}(R_n), \varphi, f)$ exhibit homomesy with average 0.



Homomesy 4?

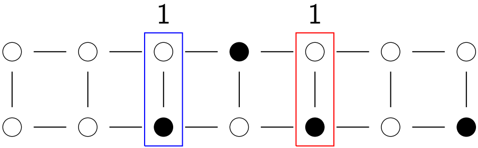
One might wonder...

Conjecture (Homomesy 4?)

For $\mathcal{I}(R_n)$ with $n \geq 3$ given and

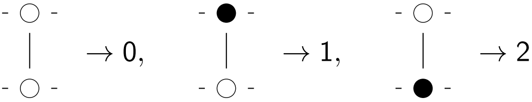
$$f(S) = \chi_{(1,3)} + \chi_{(2,3)} - (\chi_{(1,n-2)} + \chi_{(2,n-2)})$$

then the triple $(\mathcal{I}(R_n), \varphi, f)$ exhibit homomesy with average 0.

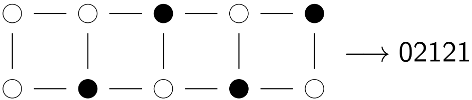


- There is a counter example in $\mathcal{I}(R_7)$; an orbit that contains 12 elements. You may use the element above to generate this counter example.

There is a bijection between $\mathcal{I}(R_n)$ and ternary strings of $0, 1$ and 2 and of length n which do not contain substrings of 11 and 22 by the following map. Given an S we map the columns of R_n like so,



For example



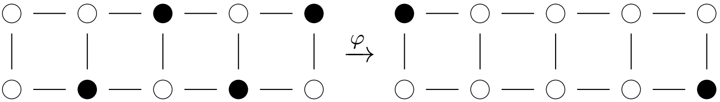
Toggling and Ternary Strings

The action φ carries over to ternary strings in a nice way.

- Let $\tau_i = \tau_{(2,i)}\tau_{(1,i)}$.
- $\varphi = \tau_{(2,n)}\tau_{(1,n)} \cdots \tau_{(2,1)}\tau_{(1,1)} = \tau_n \ldots \tau_1$.

To toggle $\tau_i = \tau_{(2,i)}\tau_{(1,i)}$ in the string just increment the ith entry modulo 3 until you get a string with no substring 11 or 22 from left to right.

$$02121 \xrightarrow{\tau_1} 12121 \xrightarrow{\tau_2} 10121 \xrightarrow{\tau_3} 10021 \xrightarrow{\tau_4} 10001 \xrightarrow{\tau_5} 10002$$



- The number of these string for $\mathcal{I}(R_n)$ is given by $a(n) = 2a(n-1) + a(n-2)$ where $a(0) = 1$ and $a(1) = 3$.

n	0	1	2	3	4	5	6	...
a(n)	1	3	7	17	41	99	239	...

- $a(n)$ is also sequence OEIS A078057 .

The Nine φ-Orbit Boards of $\mathcal{I}(R_4)$

2	1	2	1
0	0	0	2
1	2	1	0
0	0	2	1
1	0	0	2
2	1	0	0
0	2	1	2
1	0	0	0

0	1	2	1
2	0	0	2
0	1	0	0
2	0	1	2
0	2	0	0
1	0	1	2
2	0	0	0

1	2	0	1
0	0	2	0
1	0	0	1
2	1	2	0
0	0	0	1
1	2	0	2
0	0	1	0

2	0	2	1
0	1	0	2
2	0	1	0
0	2	0	1
1	0	2	0
2	1	0	1
0	2	0	2
1	0	1	0

1	2	1	2
0	0	0	0

0	0	1	2
1	2	0	0

1	0	2	1
2	1	0	2
0	2	1	0

0	1	0	1
2	0	2	0

2	0	0	1
0	1	2	0

The 41 elements of $\mathcal{I}(R_4)$ partitioned into nine orbit boards.

0-Snakes in $\mathcal{I}(R_n)$

Conjecture (Homomesy 2)

For $\mathcal{I}(R_n)$ with $n \geq 2$ given and let

$$f(S) = \chi_{(1,1)} + \chi_{(2,1)} - (\chi_{(1,n)} + \chi_{(2,n)})$$

then the triple $(\mathcal{I}(R_n), \varphi, f)$ exhibit homomesy with average 0.

We can prove Homomesy 2 similar to how we proved the theorem from path graphs. Notice how we can partition the orbit boards into snakes of zeroes. Thus we can identify zeroes in the first and last column.

2	0	2	1
0	1	0	2
2	0	1	0
0	2	0	1
1	0	2	0
2	1	0	1
0	2	0	2
1	0	1	0

1	2	0	1
0	0	2	0
1	0	0	1
2	1	2	0
0	0	0	1
1	2	0	2
0	0	1	0

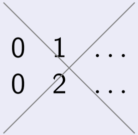
1	0	2	1
2	1	0	2
0	2	1	0

Zero Snake Lemma

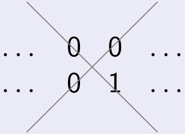
Lemma

For any orbit board in $\mathcal{I}(R_n)$:

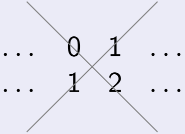
- *If a zero is the first or last column then there is no zero below.*



- *If a zero is to the left of another zero then there then it is not also above another zero*



- *If a zero is to the left of a nonzero and above a nonzero then there is a zero diagonally down right.*



Proof of Zero Snake Lemma

Proof.

- If a zero is the first or last column then there is no zero below.

The only way for a 0 to increment back around to a 0 is for it to be adjacent is a 1 and a 2 but since the 0 is on the first or last column this is impossible.

- If a zero is to the left of another zero then there then it is not also above another zero

Again the only way to increment a 0 back to a zero is for it to be adjacent to a 1 and a 2 but one of its neighbors is already a 0 so this is impossible.

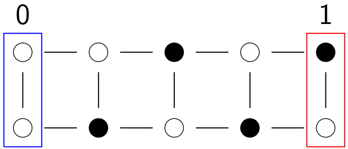
- If a zero is to the left of a nonzero and above a nonzero then there is a zero diagonally down right.

There are two cases:

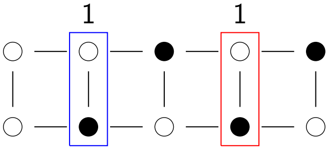
$$\begin{matrix} 0 & 1 \\ 2 & \end{matrix} \longrightarrow \begin{matrix} 0 & 1 \\ 2 & 0 \end{matrix} \qquad \begin{matrix} 0 & 2 \\ 1 & \end{matrix} \longrightarrow \begin{matrix} 0 & 2 \\ 1 & 0 \end{matrix}$$

□

Homomesy 2 Conclusion



These statements allow us to say that zero snakes will never split or end prematurely. Also that they match exactly one zero is another zero without ambiguity. Thus Homomesy 2 is proved.



Because the zeros in the second column are allowed to be stacked on top of each other we cannot use this trick to prove Homomesy 3.

Observation

An idea for how to get Homomesy 3. We should prove the following

Conjecture

Either the leftmost two columns and the rightmost two columns of a φ orbit board have the same number of 1s and 2s in the same order up to vertical shift or this is true if one side's 1s and 2s are interchanged and order is reversed.

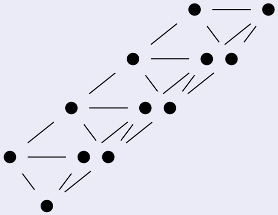
1	2	0	1
0	0	2	0
1	0	0	1
2	1	2	0
0	0	0	1
1	2	0	2
0	0	1	0

2	0	2	1
0	1	0	2
2	0	1	0
0	2	0	1
1	0	2	0
2	1	0	1
0	2	0	2
1	0	1	0

Toggling Independent Sets on $K_m \times [n]$

Instead of looking at ternary strings that do not contain substrings of 11 or 22 we can generalize to string of $\{0, 1, \ldots, m\}$ which do not contain substrings of jj for any $j = 1, \ldots, m$. Toggling at a entry would change to incriminating the entry modulo $m + 1$ until you get a string without the substring jj for all nonzero j.

Example ($K_3 \times [4]$)



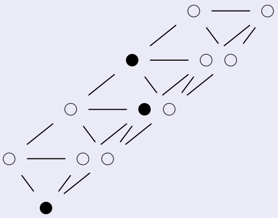
This equates to toggling independent sets on $K_m \times [n]$ where K_m is the complete graph on m vertices.

$K_m \times [n]$ Graph

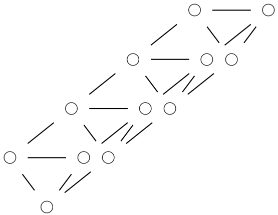
Given a map from $\{1, \ldots, m\}$ onto the vertices of K_m we can find a bijection between $\mathcal{I}(K_m \times [n])$ and m-strings.

Example ($K_3 \times [4]$)

$1320 \rightarrow$



Homomesy for $\mathcal{I}(K_m \times [n])$



Conjecture

Homomesy 1, 2, and 3 extend to $\mathcal{I}(K_m \times [n])$ *with the following tweaks: For certain distinct* $a, b \in \{1, \ldots, m\}$ *redefine the statistics as follows*

- *Homomesy 1: new* $f = \sum_{i=1}^{n-1} \chi_{(a,i)} - \sum_{i=2}^{n} \chi_{(b,i)}$
- *Homomesy 2: new* $f = \chi_{(1,1)} + \cdots + \chi_{(m,1)} - (\chi_{(1,n)} + \cdots + \chi_{(m,n)})$
- *Homomesy 3: new* $f = \chi_{(1,2)} + \cdots + \chi_{(m,2)} - (\chi_{(1,n-1)} + \cdots + \chi_{(m,n-1)})$

Example Orbit Board from $K_4 \times 6$

0	0	1	2	3	1
1	2	3	4	0	2
3	4	0	0	1	3
0	0	1	2	4	0
1	2	3	0	0	1
3	4	0	1	2	3
0	0	2	3	4	0
1	3	4	0	0	1
2	0	0	1	2	3
3	1	2	3	4	0
4	3	4	0	0	1
0	0	0	1	2	3
1	2	3	4	0	4
3	4	0	0	1	0

Example Orbit Board from $K_4 \times 6$

0	0	1	2	3	1
1	2	3	4	0	2
3	4	0	0	1	3
0	0	1	2	4	0
1	2	3	0	0	1
3	4	0	1	2	3
0	0	2	3	4	0
1	3	4	0	0	1
2	0	0	1	2	3
3	1	2	3	4	0
4	3	4	0	0	1
0	0	0	1	2	3
1	2	3	4	0	4
3	4	0	0	1	0

Example (Homomesy 1 with $a = 3$ and $b = 1$)

We find $\sum_{i=1}^{n-1} \chi_{(3,i)} = 12$ *and* $\sum_{i=2}^{n} \chi_{(1,i)} = 12$.

Example Orbit Board from $K_4 \times 6$

0	0	1	2	3	1
1	2	3	4	0	2
3	4	0	0	1	3
0	0	1	2	4	0
1	2	3	0	0	1
3	4	0	1	2	3
0	0	2	3	4	0
1	3	4	0	0	1
2	0	0	1	2	3
3	1	2	3	4	0
4	3	4	0	0	1
0	0	0	1	2	3
1	2	3	4	0	4
3	4	0	0	1	0

Example (Homomesy 1 with $a = 4$ and $b = 2$)

We find $\sum_{i=1}^{n-1} \chi_{(3,i)} = 11$ *and* $\sum_{i=2}^{n} \chi_{(1,i)} = 11$.

Example Orbit Board from $K_4 \times 6$

0	0	1	2	3	1
1	2	3	4	0	2
3	4	0	0	1	3
0	0	1	2	4	0
1	2	3	0	0	1
3	4	0	1	2	3
0	0	2	3	4	0
1	3	4	0	0	1
2	0	0	1	2	3
3	1	2	3	4	0
4	3	4	0	0	1
0	0	0	1	2	3
1	2	3	4	0	4
3	4	0	0	1	0

Example (Homomesy 2)

We find $\chi_{(1,1)} + \cdots + \chi_{(m,1)} = 10$ *and* $(\chi_{(1,n)} + \cdots + \chi_{(m,n)}) = 10$

Example Orbit Board from $K_4 \times 6$

0	0	1	2	3	1
1	2	3	4	0	2
3	4	0	0	1	3
0	0	1	2	4	0
1	2	3	0	0	1
3	4	0	1	2	3
0	0	2	3	4	0
1	3	4	0	0	1
2	0	0	1	2	3
3	1	2	3	4	0
4	3	4	0	0	1
0	0	0	1	2	3
1	2	3	4	0	4
3	4	0	0	1	0

Example (Homomesy 3)

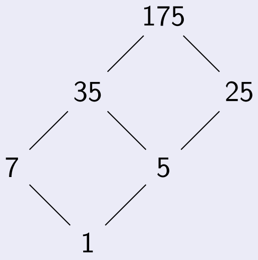
We find $\chi_{(1,2)} + \cdots + \chi_{(m,2)} = 9$ *and* $(\chi_{(1,n-1)} + \cdots + \chi_{(m,n-1)}) = 9$

Relating these Maps to Promotion of Order Ideals in Posets

- Let $(P, \leq)$ be a partially ordered set (**poset**) where P is nonempty set and $\leq$ is a relation on the elements of P which is reflexive, antisymmetric, and transitive.
- When P is finite we can represent $(P, \leq)$ by it's **Hasse diagram**.

Example (Divisors of 175)

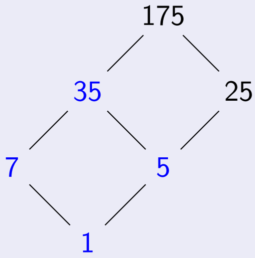
Let $P = \{x : x|175\}$ *and* $x \leq y$ *if and only if* $x|y$.



Order Ideals

- An order ideal of P is a subset I of P such that if $y \in I$ and $x < y$ then $x \in I$.

Example (An order ideal of the Divisors of 175)

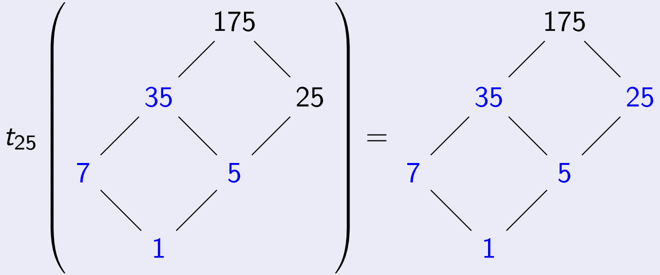


- Let $\mathcal{J}(P)$ denote the set of all order ideals of $(P, \leq)$

Toggling Order Ideals of a Posets

- Let $I \in \mathcal{J}(P)$. For all $y \in P$, let $t_y(I) : \mathcal{J}(P) \to \mathcal{J}(P)$ denote the following map
$$t_y(I) = \begin{cases} I \setminus \{y\} & \text{if } y \in I \text{ and } I \setminus y \in \mathcal{J}(P), \\ I \cup \{y\} & \text{if } y \notin I \text{ and } I \cup \{y\} \in \mathcal{J}(P), \\ I & \text{otherwise.} \end{cases}$$
Similar to independent sets, we call t_y the toggle at y.

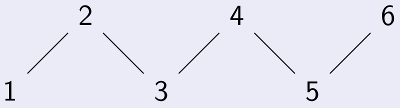
Example (Toggle at 25 in the Divisors of 175)



Zigzag Posets

- Let $Z_n = ([n], \leq)$ be the **zigzag poset** where $i \leq i + 1$ and $i \leq i - 1$ provided i is odd.

Example (Z_6)



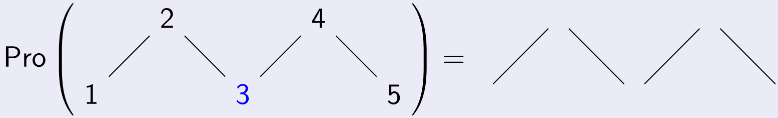
- Let $\mathsf{Pro} = t_n \cdots t_1$.

- We call this product of toggles Pro because of it's connection to rowmotion and promotion on order ideals, paved by [SW12].

Toggling the Zigzag Posets

$$t_y(I) = \begin{cases} I \setminus \{y\} & \text{if } y \in I \text{ and } I \setminus y \in \mathcal{J}(P), \\ I \cup \{y\} & \text{if } y \notin I \text{ and } I \cup \{y\} \in \mathcal{J}(P), \\ I & \text{otherwise.} \end{cases}$$

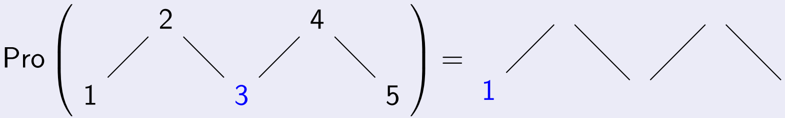
Example (Pro of an order ideal of Z_5)



Toggling the Zigzag Posets

$$t_y(I) = \begin{cases} I \setminus \{y\} & \text{if } y \in I \text{ and } I \setminus y \in \mathcal{J}(P), \\ I \cup \{y\} & \text{if } y \notin I \text{ and } I \cup \{y\} \in \mathcal{J}(P), \\ I & \text{otherwise.} \end{cases}$$

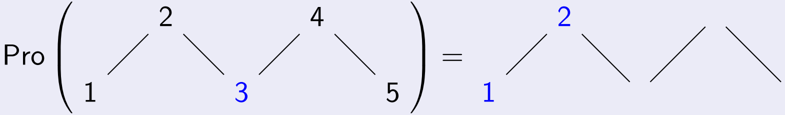
Example (Pro of an order ideal of Z_5)



Toggling the Zigzag Posets

$$t_y(I) = \begin{cases} I \setminus \{y\} & \text{if } y \in I \text{ and } I \setminus y \in \mathcal{J}(P), \\ I \cup \{y\} & \text{if } y \notin I \text{ and } I \cup \{y\} \in \mathcal{J}(P), \\ I & \text{otherwise.} \end{cases}$$

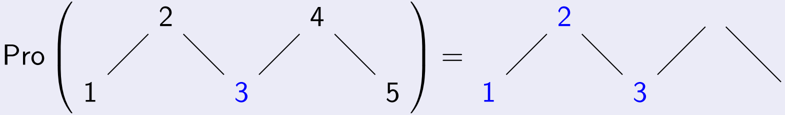
Example (Pro of an order ideal of Z_5)



Toggling the Zigzag Posets

$$t_y(I) = \begin{cases} I \setminus \{y\} & \text{if } y \in I \text{ and } I \setminus y \in \mathcal{J}(P), \\ I \cup \{y\} & \text{if } y \notin I \text{ and } I \cup \{y\} \in \mathcal{J}(P), \\ I & \text{otherwise.} \end{cases}$$

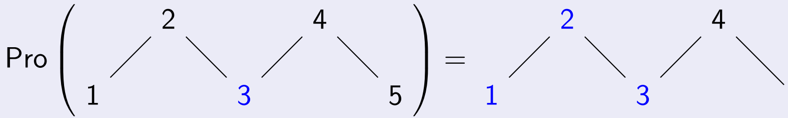
Example (Pro of an order ideal of Z_5)



Toggling the Zigzag Posets

$$t_y(I) = \begin{cases} I \setminus \{y\} & \text{if } y \in I \text{ and } I \setminus y \in \mathcal{J}(P), \\ I \cup \{y\} & \text{if } y \notin I \text{ and } I \cup \{y\} \in \mathcal{J}(P), \\ I & \text{otherwise.} \end{cases}$$

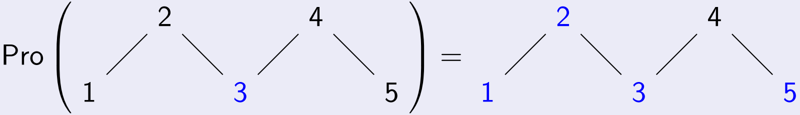
Example (Pro of an order ideal of Z_5)



Toggling the Zigzag Posets

$$t_y(I) = \begin{cases} I \setminus \{y\} & \text{if } y \in I \text{ and } I \setminus y \in \mathcal{J}(P), \\ I \cup \{y\} & \text{if } y \notin I \text{ and } I \cup \{y\} \in \mathcal{J}(P), \\ I & \text{otherwise.} \end{cases}$$

Example (Pro of an order ideal of Z_5)



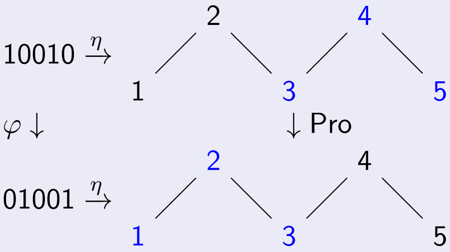
Equivariant Bijection

Theorem ([JR18, Prop. 5.4,5.6])

There exists an bijection $\eta : \mathcal{I}(P_n) \to \mathcal{J}(Z_n)$ such that $\mathrm{Pro} \circ \eta = \eta \circ \varphi$, making η an ***equivariant*** *bijection.*

- The bijection η is given by mapping the vertices of S to the elements of I.
  - If j is odd then jth entry being 1 means $j \notin I$ otherwise $j \in I$
  - If j is even then jth entry being 1 means $j \in I$ otherwise $j \notin I$

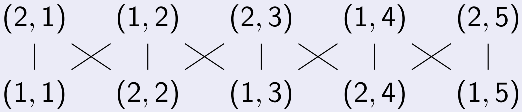
Example



The W_n Poset

- Let $W_n = ([2] \times [n], \leq)$ be the poset where
  - $(1, i) \leq (1, i+1)$ and $(1, i) \leq (1, i-1)$ provided i is odd,
  - $(2, i) \leq (2, i+1)$ and $(2, i) \leq (2, i-1)$ provided i is even,
  - $(1, i) \leq (2, i)$ when i is odd,
  - and $(2, i) \leq (1, i)$ when i is even.

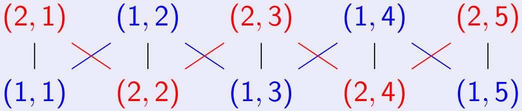
Example (W_5)



The W_n Poset

- Let $W_n = ([2] \times [n], \leq)$ be the poset where
  - $(1, i) \leq (1, i+1)$ and $(1, i) \leq (1, i-1)$ provided i is odd,
  - $(2, i) \leq (2, i+1)$ and $(2, i) \leq (2, i-1)$ provided i is even,
  - $(1, i) \leq (2, i)$ when i is odd,
  - and $(2, i) \leq (1, i)$ when i is even.

Example (W_5)

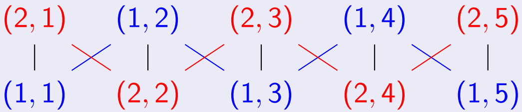


- We can see two copies of Z_5 stacked on top of each other in W_5, albeit with one copy of Z_5 flipped upside down.

The W_n Poset

- Let $W_n = ([2] \times [n], \leq)$ be the poset where
  - $(1,i) \leq (1,i+1)$ and $(1,i) \leq (1,i-1)$ provided i is odd,
  - $(2,i) \leq (2,i+1)$ and $(2,i) \leq (2,i-1)$ provided i is even,
  - $(1,i) \leq (2,i)$ when i is odd,
  - and $(2,i) \leq (1,i)$ when i is even.

Example (W_5)



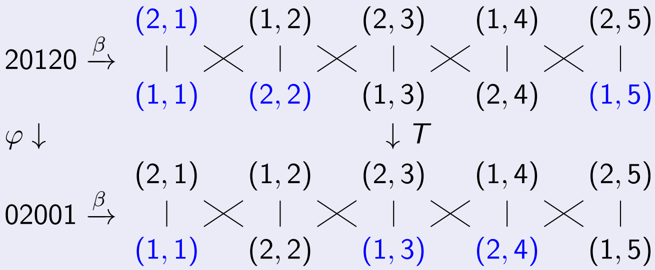
Conjecture

Let $T = t_{(2,n)} \cdots t_{(2,1)} t_{(1,n)} \cdots t_{(1,1)}$. There exists an bijection $\beta : \mathcal{I}(R_n) \to \mathcal{J}(W_n)$ such that $T \circ \beta = \beta \circ \varphi$, making β an ***equivariant*** *bijection.*

Equivariant Bijection?

- This candidate bijection β is given by mapping the vertices of S to the elements of I.
  - If j is odd then jth entry being 1 means $(1, j) \notin I$ otherwise $(1, j) \in I$
  - If j is odd then jth entry being 2 means $(2, j) \in I$ otherwise $(2, j) \notin I$
  - If j is even then jth entry being 1 means $(1, j) \in I$ otherwise $(1, j) \notin I$
  - If j is even then jth entry being 2 means $(2, j) \notin I$ otherwise $(2, j) \in I$

Example



The End is Nigh

Thank You!

References and Citations

[SW11] Jessica Striker and Nathan Williams, *Promotion and Rowmotion*, Europ. J. of Combin. 33 (2012), 1919–1942,

[PR15] James Propp and Tom Roby, *Homomesy in products of two chains*, Electronic J. Combin. **22(3)** (2015), #P3.4, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p4.

[R16] Tom Roby, *Dynamical algebraic combinatorics and the homomesy phenomenon* in A. Beveridge, et. al., Recent Trends in Combinatorics, IMA Volumes in Math. and its Appl., **159** (2016), 619–652.

[JR18] Michael Joseph and Tom Roby, *Toggling Independent Sets of a Path Graph*, Electronic Journal of Combin., **25(1)**, 2018, #P1.18.